ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in [0,1]$ and $a+b+c=2$, then prove that :

$$a^4+b^4+c^4+\frac{11}{2}abc \le \frac{5}{2}$$

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$$a^4+b^4+c^4+\frac{11}{2}abc-\frac{5}{2}$$
$$=a^4+(b+c)^4-2bc(2(b+c)^2-bc)+\frac{11}{2}abc-\frac{5}{2}\overset{?}{\le}0$$
$$\overset{a+b+c=2}{\Leftrightarrow} a^4+(2-a)^4-4bc(2-a)^2+2b^2c^2+\frac{11}{2}abc-\frac{5}{2}\overset{?}{\le}0$$
$$\Leftrightarrow 4b^2c^2+(11a-8(2-a)^2)bc+2a^4+2(2-a)^4-5\overset{?}{\le}0 \quad (1)$$

Now, LHS of (1) is a quadratic polynomial in bc with discriminant $\delta =$
$$(11a-8(2-a)^2)^2-16(2a^4+2(2-a)^4-5)$$
$$=\frac{(4509-2160a)(5a-4)^2+1856-1080a}{125}\overset{a\le 1}{>}0$$

and so, in order to prove : (1), it suffices to prove :

$$bc \overset{(i)}{\le} \frac{8a^2-43a+32+\sqrt{\delta}}{8} \text{ AND } bc \overset{(ii)}{\ge} \frac{8a^2-43a+32-\sqrt{\delta}}{8}$$

$\because (b-1)(c-1)\ge 0 \overset{a+b+c=2}{\therefore} bc \ge 2-a-1 \therefore$ in order to prove (ii),

it suffices to prove : $8-8a\overset{?}{\ge}8a^2-43a+32-\sqrt{\delta}$

$$\Leftrightarrow \sqrt{\delta}\overset{?}{\ge}8a^2-35a+24 \quad (*)$$

and it's trivially true when : $8a^2-43a+32\le 0$

and so, we now focus on the case when : $8a^2-43a+32>0$ and then : $(*)\Leftrightarrow$

$$(11a-8(2-a)^2)^2-16(2a^4+2(2-a)^4-5)\overset{?}{\ge}(8a^2-35a+24)^2$$
$$\Leftrightarrow 16(2a-1)^2\big(1+a(1-a)\big)\overset{?}{\ge}0 \to \text{true} \because 0\le a\le 1 \Rightarrow (*)\Rightarrow \text{(ii) is true}$$

and again, via AM $-$ GM, $bc\le\frac{(2-a)^2}{4}\overset{?}{\le}\frac{8a^2-43a+32+\sqrt{\delta}}{8}$

$$\Leftrightarrow 6a^2-35a+24+\sqrt{\delta}\overset{?}{\ge}0 \quad (**)$$

and it's trivially true when : $6a^2-35a+24\ge 0$

and so, we now focus on the case when : $6a^2-35a+24<0$ and then : $(**)\Leftrightarrow$

$$(11a-8(2-a)^2)^2-16(2a^4+2(2-a)^4-5)\overset{?}{\ge}(6a^2-35a+24)^2$$
$$\Leftrightarrow 4(1-a)(9a^3+8a^2+4(a-1)^2)\overset{?}{\ge}0 \to \text{true} \because 0\le a\le 1 \Rightarrow (**)\Rightarrow \text{(i) is true}$$

and so, both (i) and (ii) are true $\Rightarrow (1)$ is true

ROMANIAN MATHEMATICAL MAGAZINE

$$\therefore a^4+b^4+c^4+\frac{11}{2}abc\le\frac{5}{2}\ \forall a,b,c\in[0,1]\ \big|\ a+b+c=2,$$

$''='' \text{ iff } \left(a=\frac{1}{2},b=1,c=\frac{1}{2}\right) \text{ or } \left(a=\frac{1}{2},b=\frac{1}{2},c=1\right) \text{ or } \left(b=\frac{1}{2},c=1,a=\frac{1}{2}\right)$
$\text{or } \left(b=\frac{1}{2},c=\frac{1}{2},a=1\right) \text{ or } \left(c=\frac{1}{2},a=1,b=\frac{1}{2}\right) \text{ or } \left(c=\frac{1}{2},a=\frac{1}{2},b=1\right)$

(QED)