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If $a,b,c>0$, then prove that :

$$\frac{a(a+b)}{(2a+b)(b+c)}+\frac{b(b+c)}{(2b+c)(c+a)}+\frac{c(c+a)}{(2c+a)(a+b)}\geq 1$$

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$$\frac{a(a+b)}{(2a+b)(b+c)}+\frac{b(b+c)}{(2b+c)(c+a)}+\frac{c(c+a)}{(2c+a)(a+b)}$$
$$=\sum_{cyc}\frac{a}{\left(\frac{a}{a+b}+1\right)(b+c)}=\sum_{cyc}\frac{a^2}{ab+ac+\frac{a^2(b+c)}{a+b}}$$
$$\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{2\sum_{cyc}ab+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a^2}{a+b}\right)-\sum_{cyc}\frac{a^3}{a+b}}\overset{?}{\geq}1$$
$$\Leftrightarrow\left(\sum_{cyc}a\right)^2+\sum_{cyc}\frac{a^3}{a+b}\overset{?}{\geq}2\sum_{cyc}ab+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a(a+b-b)}{a+b}\right)$$
$$=2\sum_{cyc}ab+\left(\sum_{cyc}a\right)^2-\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{ab}{a+b}\right)$$
$$\Leftrightarrow\sum_{cyc}\frac{a^3}{a+b}+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{ab}{a+b}\right)\overset{?}{\geq}2\sum_{cyc}ab\ (*)$$

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius
$=s,R,r$ (say); and then : $\sum_{cyc}a=s\rightarrow(1), abc=r^2s\rightarrow(2)$,

$$\sum_{cyc}ab=4Rr+r^2\rightarrow(3),\sum_{cyc}a^2=s^2-8Rr-2r^2\rightarrow(4)$$

$$\text{Now, }\sum_{cyc}\frac{a^3}{a+b}+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{ab}{a+b}\right)$$
$$=\sum_{cyc}\frac{a^4}{a^2+ab}+\left(\sum_{cyc}a\right)\left(\sum_{cyc}\frac{a^2b^2}{a^2b+ab^2}\right)\overset{\text{Bergstrom}}{\geq}$$

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$$\frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} a^2+\sum_{cyc} ab}+\frac{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)^2}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-3abc}\overset{?}{\geq} 2\sum_{cyc} ab$$

$$\Leftrightarrow \frac{\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-3abc\right)\left(\sum_{cyc} a^2\right)^2+\left(\sum_{cyc} a^2+\sum_{cyc} ab\right)\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)^2}{\left(\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)-3abc\right)\left(\sum_{cyc} a^2+\sum_{cyc} ab\right)}$$

$$\overset{?}{\geq} 2\sum_{cyc} ab \overset{\text{via } (1),(2),(3) \text{ and } (4)}{\Leftrightarrow} (s(4Rr+r^2)-3r^2s)(s^2-8Rr-2r^2)^2$$

$$+s(s^2-4Rr-r^2)(4Rr+r^2)^2\overset{?}{\geq} 2(4Rr+r^2)(s(4Rr+r^2)-3r^2s)(s^2-4Rr-r^2)$$

$$\Leftrightarrow (4R-2r)s^4-r(80R^2-32Rr-13r^2)s^2$$

$$+r^2(320R^3-48R^2r-84Rr^2-13r^3)\boxed{\overset{?}{\geq}}0 \;\&\; \because P=(4R-2r)(s^2-16Rr+5r^2)^2$$

$$\boxed{(**)}$$

$$+r(48R^2-72Rr+33r^2)(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**)\overset{?}{\geq} P \Leftrightarrow 2t^3-9t^2+12t-4\overset{?}{\geq}0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (2t-1)(t-2)^2\overset{?}{\geq}0\to \text{true} \because t\overset{\text{Euler}}{\geq} 2r\Rightarrow (**)\Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a(a+b)}{(2a+b)(b+c)}+\frac{b(b+c)}{(2b+c)(c+a)}+\frac{c(c+a)}{(2c+a)(a+b)}\geq 1\ \forall a,b,c>0,$$

$$'' ='' \text{ iff } a=b=c \text{ (QED)}$$