ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$, then prove that :

$$a^3+b^3+c^3+6abc\geq 9\cdot\frac{a^2+b^2+c^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); and then : $abc=r^2s\rightarrow(1)$, $\sum_{cyc} ab=4Rr+r^2\rightarrow(2)$,

$\sum_{cyc} a^2=s^2-8Rr-2r^2\rightarrow(3)$ and $\sum_{cyc} a^3=s(s^2-12Rr)\rightarrow(4)$ and so,

$$a^3+b^3+c^3+6abc\geq 9.\frac{a^2+b^2+c^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \overset{\text{via }(1),(2),(3)\text{ and }(4)}{\Leftrightarrow}$$

$$(s(s^2-12Rr)+6r^2s)(4Rr+r^2)\geq 9r^2s(s^2-8Rr-2r^2)$$
$$\Leftrightarrow (R-2r)s^2\geq r(R-2r)(12R+3r)$$
$$\Leftrightarrow (R-2r)\left((s^2-16Rr+5r^2)+4r(R-2r)\right)\geq 0\rightarrow \text{true via Gerretsen and Euler}$$

$$\therefore a^3+b^3+c^3+6abc\geq 9.\frac{a^2+b^2+c^2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\forall\, a,b,c>0,$$

$''=''$ iff $a=b=c=1$ (QED)