ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$a^2+b^2+c^2\geq a^2b^2c^2(a^3+b^3+c^3)$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0$,

$y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y$

$\Rightarrow x,y,z$ form sides of $a$ triangle XYZ with semiperimeter, circumradius $a$nd inradius

$=s,R,r$ (say); then : $\sum_{cyc} a=s\rightarrow(1), abc=r^2s\rightarrow(2)$,

$$\sum_{cyc} a^2=s^2-8Rr-2r^2\rightarrow(3) \text{ and } \sum_{cyc} a^3=s(s^2-12Rr)\rightarrow(4)$$

$$\text{and so, } a^2+b^2+c^2\geq a^2b^2c^2(a^3+b^3+c^3) \overset{a+b+c=3}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a\right)^7\geq 2187a^2b^2c^2\left(\sum_{cyc} a^3\right)$$

$$\overset{\text{via } (1),(2),(3) \text{ and } (4)}{\Leftrightarrow} (s^2-8Rr-2r^2)s^7\geq 2187r^4s^3(s^2-12Rr)$$

$$\Leftrightarrow s^4(s^2-8Rr-2r^2)\overset{(*)}{\geq} 2187r^4(s^2-12Rr)$$

$$\text{and } \because P=(s^2-16Rr+5r^2)^3+(40Rr-17r^2)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\geq P$

$$\Leftrightarrow (128R^2-116Rr-523r^2)s^2\overset{?}{\geq}r(1536R^3-1728R^2r-5931Rr^2-75r^3)\quad (**)$$

**Case 1** $128R^2-116Rr-523r^2\geq 0$ and then : LHS of $(**)\overset{\text{Gerretsen}}{\geq}$

$$(128R^2-116Rr-523r^2)(16Rr-5r^2)\overset{?}{\geq}r\left(\begin{matrix}1536R^3-1728R^2r-5931Rr^2\\-75r^3\end{matrix}\right)$$

$$\Leftrightarrow 512t^3-768t^2-1857t+2690\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\big((t-2)(512t+1280)+1215\big)\overset{?}{\geq}0\rightarrow \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow (**) \text{ is true}$$

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**Case 2** $128R^2-116Rr-523r^2<0$ *and then* : LHS of $(**)\overset{\text{Gerretsen}}{\geq}$

$$(128R^2-116Rr-523r^2)(4R^2+4Rr+3r^2)\overset{?}{\geq} r\begin{pmatrix}1536R^3-1728R^2r\\-5931Rr^2-75r^3\end{pmatrix}$$

$$\Leftrightarrow 512t^4-1488t^3-444t^2+3491t-1494\overset{?}{\geq}0$$

$$\Leftrightarrow (t-2)\Big((t-2)(512t^2+560t-252)+243\Big)\overset{?}{\geq}0\rightarrow \text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (**)\text{ is true}\therefore \text{combining both cases},(**)\Rightarrow(*)\text{ is true}\forall\Delta XYZ$$

$$\therefore a^2+b^2+c^2\geq a^2b^2c^2(a^3+b^3+c^3)\ \forall a,b,c>0\Big|\sum_{cyc}a=3,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$