ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then prove that :

$$(1-abc)\left(a^2+b^2+c^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}\right)\geq 0$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

and then : $\sum_{cyc} a=s\rightarrow(1),, abc=r^2s\rightarrow(2), \sum_{cyc} ab=4Rr+r^2\rightarrow(3),$

$\sum_{cyc} a^2=s^2-8Rr-2r^2\rightarrow(4)$ and $\sum_{cyc} a^2b^2=r^2((4R+r)^2-2s^2)\rightarrow(5)$ and so,

$$(1-abc)\left(\sum_{cyc} a^2-\sum_{cyc}\frac{1}{a^2}\right)\geq 0 \Leftrightarrow \sum_{cyc} a^2-\sum_{cyc}\frac{1}{a^2}\geq abc\sum_{cyc} a^2-abc\sum_{cyc}\frac{1}{a^2}$$

$$\overset{abc\sum_{cyc}a=\sum_{cyc}ab}{\Leftrightarrow}\left(\frac{1-a^2b^2c^2}{1+abc}\right)\left(\sum_{cyc} a^2-\frac{\sum_{cyc} a^2b^2}{a^2b^2c^2}\cdot\left(\frac{abc\sum_{cyc} a}{\sum_{cyc} ab}\right)^2\right)\geq 0$$

$$\overset{abc\sum_{cyc}a=\sum_{cyc}ab}{\Leftrightarrow}\frac{\left(\left(\frac{abc\sum_{cyc} a}{\sum_{cyc} ab}\right)^3-a^2b^2c^2\right)\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2-\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2\right)}{(1+abc)\left(\sum_{cyc} ab\right)^2}$$

$$\geq 0 \Leftrightarrow \frac{a^2b^2c^2}{(1+abc)\left(\sum_{cyc} ab\right)^5}\cdot\left(abc\left(\sum_{cyc} a\right)^3-\left(\sum_{cyc} ab\right)^3\right)\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)^2-\left(\sum_{cyc} a^2b^2\right)\left(\sum_{cyc} a\right)^2\right)\geq 0$$

$$\overset{via\ (1),(2),(3),(4)\ and\ (5)}{\Leftrightarrow}(r^2s^4-r^3(4R+r)^3)\begin{pmatrix}(s^2-8Rr-2r^2).r^2(4R+r)^2\\-r^2s^2((4R+r)^2-2s^2)\end{pmatrix}\geq 0$$

$$\Leftrightarrow 2r^4(s^4-r(4R+r)^3)(s^4-r(4R+r)^3)\geq 0 \Leftrightarrow 2r^4(s^4-r(4R+r)^3)^2\geq 0$$

$$\rightarrow \text{true} \therefore (1-abc)\left(a^2+b^2+c^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}\right)\geq 0$$

$$\forall a,b,c>0\,|\,a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},''='' \text{ iff } a=b=c=1 \text{ (QED)}$$