ROMANIAN MATHEMATICAL MAGAZINE

If $x, y \in [0,1]$, then prove that :

$$8(x+y-1)^2 - 9xy(x+y-1) + xy \geq 0$$

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LHS is a quadratic polynomial in $(x+y-1)$ with discriminant, $\delta = 81x^2y^2 - 32xy$ and if : $xy \leq \frac{32}{81}$, then : $\delta \leq 0 \Rightarrow \text{LHS} \geq 0$ and we now focus on : $xy > \frac{32}{81}$ and, in order to prove : $\text{LHS} \geq 0$, it suffices to prove :

$$x+y-1 \overset{?}{\geq} \frac{9xy + \sqrt{81x^2y^2 - 32xy}}{16}$$

and to prove it, it suffices to prove :

$$32\sqrt{xy} - 16 - 9xy \boxed{\overset{?}{\geq}\ (*)} \sqrt{81x^2y^2 - 32xy} \quad \left(\because x+y \overset{A-G}{\geq} 2\sqrt{xy}\right)$$

Now, $9\mu^2 - 32\mu + 16 \overset{?}{<} 0 \left(\mu = \sqrt{xy}\right) \Leftrightarrow \mu < \frac{32+\sqrt{448}}{18}$ and $\mu > \frac{32-\sqrt{448}}{18}$

$\rightarrow$ both are true $\because \mu \overset{x,y \in [0,1]}{\leq} 1 < \frac{32+\sqrt{448}}{18}$ and $\mu \overset{xy > \frac{32}{81}}{>} \frac{4\sqrt{2}}{9} > \frac{32-\sqrt{448}}{18}$

$\therefore 9xy - 32\sqrt{xy} + 16 < 0 \Rightarrow 32\sqrt{xy} - 16 - 9xy > 0$ and so, $(*)$

$\Leftrightarrow (32\mu - 16 - 9\mu^2)^2 \overset{?}{\geq} 81\mu^4 - 32\mu^2 \Leftrightarrow 64(1-\mu)(3\mu-2)^2 \overset{?}{\geq} 0 \rightarrow$ true

$\because x, y \in [0,1] \Rightarrow 1 \geq \mu \Rightarrow (*)$ is true $\therefore 8(x+y-1)^2 - 9xy(x+y-1) + xy \geq 0$

$\forall\, x, y \in [0,1]$, $''='' $ iff $x = y = 1$ (QED)