ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2=3$, then prove that :

$$2(a^4+b^4+c^4)+\frac{9a^2b^2c^2}{ab+bc+ca}\geq 9$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

and then : $\sum_{cyc} a=s, abc=r^2s, \sum_{cyc} ab=4Rr+r^2, \sum_{cyc} a^2=s^2-8Rr-2r^2$ and

$\sum_{cyc} a^2b^2=r^2((4R+r)^2-2s^2)$ and so, $2(a^4+b^4+c^4)+\frac{9a^2b^2c^2}{ab+bc+ca}\geq 9$

$$\overset{3=a^2+b^2+c^2}{\Leftrightarrow} 2\left((s^2-8Rr-2r^2)^2-2r^2((4R+r)^2-2s^2)\right)(4Rr+r^2)+9r^4s^2$$

$$-(4Rr+r^2)(s^2-8Rr-2r^2)^2\geq 0 \Leftrightarrow (4R+r)s^2 \overset{(*)}{\geq} r(64R^2-13r^2)$$

$$\text{Now, } (4R+r)s^2 \overset{\text{Rouche}}{\geq} (4R+r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\geq} r(64R^2-13r^2)$$

$$\Leftrightarrow 2(R-2r)(4R^2-3Rr-3r^2) \overset{?}{\geq} 2(R-2r)(4R+r).\sqrt{R^2-2Rr}$$

and to prove it, it suffices to prove : $(4R^2-3Rr-3r^2)^2 \overset{?}{>} (R^2-2Rr)(4R+r)^2$

$$\Leftrightarrow r^3(20R+9r) \overset{?}{>} 0 \rightarrow \text{true} \Rightarrow (*) \text{ is true} 2(a^4+b^4+c^4)+\frac{9a^2b^2c^2}{ab+bc+ca}\geq 9$$

$\forall a,b,c>0 \,|\, a^2+b^2+c^2=3,'' ='' $ iff $a=b=c=1$ (QED)