ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in [0,2]$ and $a+b+c=3$, then prove that :

$$a^3+b^3+c^3+abc \le 9$$

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$$\text{Either } b \le 2, c < 2 \text{ or } c \le 2, b < 2 \therefore (b-2)(c-2) \ge 0$$

$$\overset{a+b+c=3}{\Rightarrow} bc \ge 2(3-a)-4 \Rightarrow bc \overset{(1)}{\ge} 2-2a$$

$$\text{Now, } a^3+b^3+c^3+abc \le 9 \overset{a+b+c=3}{\Leftrightarrow} 3abc+3\left(9-3\sum_{cyc} ab\right)+abc \le 9$$

$$\overset{a+b+c=3}{\Leftrightarrow} 4abc+18-9\big(bc+a(3-a)\big) \le 0 \Leftrightarrow (9-4a).bc \ge 18-9a(3-a)$$

$$\Leftrightarrow bc \overset{(*)}{\ge} \frac{9(a-1)(a-2)}{9-4a} \left(\because 9-4a \overset{a \le 2}{\ge} 1 > 0\right)$$

$$\text{Now, let us assume that : } bc < \frac{9(a-1)(a-2)}{9-4a} \text{ and then, since } bc \ge 0$$

$$\therefore \frac{9(a-1)(a-2)}{9-4a} > 0 \Rightarrow a-1 < 0 \left(\begin{matrix}\because a-2 \le 0 \text{ and more specifically,} \\ \text{in this case : } a-2 < 0\end{matrix}\right)$$

$$\Rightarrow a < 1 \to (m) \text{ and also, via } (1), 2-2a \le bc < \frac{9(a-1)(a-2)}{9-4a}$$

$$\Leftrightarrow (a-1)\left(\frac{9(a-2)}{9-4a}+2\right) > 0 \Leftrightarrow \frac{a(a-1)}{9-4a} > 0 \Rightarrow a > 1 \to (n)$$

$$(\because a \ge 0 \text{ and more specifically, in this case : } a > 0) \text{ and } (m), (n) \Rightarrow$$

our assumption leads to a contradiction and hence, our assumption is incorrect

$$\Rightarrow (*) \text{ is true } \therefore a^3+b^3+c^3+abc \le 9 \ \forall \ a, b, c \in [0,2] \ \big| \ a+b+c=3,$$

$$'' = '' \text{ iff } (a=1, b=0, c=2) \text{ or } (a=1, b=2, c=0) \text{ or } (b=1, c=0, a=2)$$

$$\text{or } (b=1, c=2, a=0) \text{ or } (c=1, a=0, b=2) \text{ or } (c=1, a=2, b=0) \text{ (QED)}$$