ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then prove that :

$$\frac{1}{1+2ab}+\frac{1}{1+2bc}+\frac{1}{1+2ca}\geq 1$$

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$$\frac{1}{1+2ab}+\frac{1}{1+2bc}+\frac{1}{1+2ca}\geq 1\Leftrightarrow 1+\sum_{cyc}ab\geq 4a^2b^2c^2$$

$$\overset{\sum_{cyc}a=\frac{\sum_{cyc}ab}{abc}}{\Leftrightarrow}\left(\frac{abc\left(\sum_{cyc}a\right)}{\sum_{cyc}ab}\right)^3+\left(\frac{abc\left(\sum_{cyc}a\right)}{\sum_{cyc}ab}\right)^2\left(\sum_{cyc}ab\right)\overset{(*)}{\geq}4a^2b^2c^2 \text{ and}$$

assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

and then : $\sum_{cyc}a=s, abc=r^2s$ and $\sum_{cyc}ab=4Rr+r^2$ and so,

$$(*)\Leftrightarrow s^4+(4R+r)^2s^2\overset{(**)}{\geq}4r(4R+r)^3$$

$$\text{Now, LHS of } (**)\overset{\text{Gerretsen}}{\geq}((4R+r)^2+16Rr-5r^2)s^2\overset{\text{Gerretsen}}{\geq}$$

$$(16R^2+24Rr-4r^2)(16Rr-5r^2)\overset{?}{\geq}4r(4R+r)^3\Leftrightarrow 8r^2(14R-r)(R-2r)\overset{?}{\geq}0$$

$$\rightarrow \text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow (**)\Rightarrow (*) \text{ is true}\therefore \frac{1}{1+2ab}+\frac{1}{1+2bc}+\frac{1}{1+2ca}\geq 1$$

$$\forall a,b,c>0\,|\,a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},''='' \text{ iff } a=b=c=1 \text{ (QED)}$$