ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then prove that :

$$(a+b+c)^2+18abc\geq 27$$

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$$\sum_{cyc}a=\sum_{cyc}\frac{1}{a}\overset{\text{Bergstrom}}{\geq}\frac{9}{\sum_{cyc}a}\Rightarrow\sum_{cyc}a\geq 3 \text{ and if } abc>1, \text{ then :}$$

$$\text{LHS}>9+18=27 \text{ and we now focus on : } abc\leq 1 \text{ and then :}$$

$$\sum_{cyc}a=\frac{\sum_{cyc}ab}{abc}\geq\sum_{cyc}ab\Rightarrow \text{LHS}\geq\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+18abc\overset{?}{\geq}27$$

$$\sum_{cyc}a=\frac{\sum_{cyc}ab}{abc}=\left(\frac{27abc\left(\sum_{cyc}a\right)}{\sum_{cyc}ab}\right)^{\frac{3}{2}}$$

$$\Leftrightarrow\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)+18abc\right)^2\overset{?}{\geq}\overset{(*)}{\text{ALS}}\frac{729\left(abc\left(\sum_{cyc}a\right)\right)^3}{\left(\sum_{cyc}ab\right)^3} \text{ and assigning}$$

$$b+c=x,c+a=y,a+b=z\Rightarrow x+y-z=2c>0,y+z-x=2a>0$$
$$\text{and } z+x-y=2b>0\Rightarrow x+y>z,y+z>x,z+x>y\Rightarrow x,y,z \text{ form sides}$$
$$\text{of } a \text{ triangle with semiperimeter, circumradius and inradius} = s,R,r \text{ (say);}$$
$$\text{and then : } \sum_{cyc}a=s,abc=r^2s \text{ and } \sum_{cyc}ab=4Rr+r^2 \text{ and so, } (*)\Leftrightarrow$$

$$(s(4Rr+r^2)+18r^2s)^2\overset{?}{\geq}\frac{729(r^2s^2)^3}{(4Rr+r^2)^3}\Leftrightarrow(4R+19r)^2(4R+r)^3\overset{?}{\geq}\overset{(**)}{\text{ALS}}729rs^4$$

$$\text{and } 729rs^4\overset{\text{Gerretsen}}{\leq}729r(4R^2+4Rr+3r^2)^2\overset{?}{\leq}\text{LHS of } (**)$$

$$\Leftrightarrow 256t^5-292t^4+1816t^3-2498t^2-3253t-1550\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(256t^4+220t^3+2256t^2+2014t+775)\overset{?}{\geq}0\rightarrow\text{true}\because t\overset{\text{Euler}}{\geq}2$$

$$\Rightarrow (**)\Rightarrow (*) \text{ is true } \therefore \text{ combining both cases, } (a+b+c)^2+18abc\geq 27$$

$$\forall a,b,c>0\ \Big|\ \sum_{cyc}a=\sum_{cyc}\frac{1}{a},''='' \text{ iff } a=b=c=1 \text{ (QED)}$$