ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>\frac{1}{2}$, then prove that :

$$\frac{1}{2a-1}+\frac{1}{2b-1}+\frac{1}{2c-1}+\frac{4ab}{ab+1}+\frac{4bc}{bc+1}+\frac{4ca}{ca+1}\geq 9$$

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Let $2a-1=x, 2b-1=y, 2c-1=z$ and then :

$$\frac{1}{2}\left(\frac{1}{2a-1}+\frac{1}{2b-1}\right)+\frac{4ab}{ab+1}-3=\frac{x+y}{2xy}+\frac{4(x+1)(y+1)}{(x+1)(y+1)+4}-3$$

$$=\frac{m^2+3mn+5m+2n^2-22n}{2xy\left((x+1)(y+1)+4\right)}\overset{?}{\geq}0\ (m=x+y, n=xy)$$

$$\Leftrightarrow m^2+(3n+5)m+2n^2-22n\overset{?}{\geq}0$$

$$\Leftrightarrow m\overset{?}{\geq}\frac{-(3n+5)+\sqrt{(3n+5)^2-4(2n^2-22n)}}{2}\ \text{and}\ \because m\overset{A-G}{\geq}2\sqrt{n}$$

$$\therefore \text{it suffices to prove : } 4p+3p^2+5\overset{?}{\geq}\sqrt{p^4+118p^2+25}\ (p=\sqrt{n})$$

$$\Leftrightarrow (4p+3p^2+5)^2-p^4-118p^2-25\overset{?}{\geq}0\Leftrightarrow 8p(p+5)(p-1)^2\overset{?}{\geq}0\rightarrow \text{true}$$

$$\therefore \frac{1}{2}\left(\frac{1}{2a-1}+\frac{1}{2b-1}\right)+\frac{4ab}{ab+1}\geq 3\ \text{and analogs}$$

$$\therefore \frac{1}{2a-1}+\frac{1}{2b-1}+\frac{1}{2c-1}+\frac{4ab}{ab+1}+\frac{4bc}{bc+1}+\frac{4ca}{ca+1}\geq 9$$

$$\forall a,b,c>\frac{1}{2}, ''='' \text{ iff } a=b=c=1\ (\text{QED})$$