ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2+abc=4$, then prove that :

$$a\left(1+c\sqrt{b}\right)+b\left(1+a\sqrt{c}\right)+c\left(1+b\sqrt{a}\right)\le 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+b^2+c^2+abc=4\Rightarrow a^2+a.bc+(b^2+c^2-4)=0$$

$$\Rightarrow a=\frac{-bc\pm\sqrt{b^2c^2-4(b^2+c^2-4)}}{2}=\frac{-bc\pm\sqrt{(4-b^2)(4-c^2)}}{2}$$

$$\therefore a=\frac{-bc+\sqrt{(4-b^2)(4-c^2)}}{2}\ (\because a>0)\Rightarrow 2-a$$

$$=\frac{8+2bc-2\sqrt{(4-b^2)(4-c^2)}}{4}\overset{A-G}{\ge}\frac{8+2bc-(8-b^2-c^2)}{4}=\frac{(b+c)^2}{4}$$

$$\Rightarrow\sqrt{2-a}\ge\frac{b+c}{2}\overset{A-G}{\Rightarrow}\frac{2-a+1}{2}\ge\frac{b+c}{2}\Rightarrow 3\overset{(1)}{\ge}\sum_{cyc}a\ \text{and LHS}=$$

$$\sum_{cyc}a+\sum_{cyc}(bc.\sqrt{a})\overset{\text{Chebyshev}}{\le}\sum_{cyc}a+\frac{1}{3}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\sqrt{a}\right)\overset{CBS}{\le}$$

$$\sum_{cyc}a+\frac{1}{9}\left(\sum_{cyc}a\right)^2.\sqrt{3\sum_{cyc}a}\overset{\text{via }(1)}{\le}3+3=6$$

$$\forall a,b,c>0\,|\,a^2+b^2+c^2+abc=4,''='' \text{ iff } a=b=c=1\ (\text{QED})$$