# ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$ $and$ $(3x+1)(3y+1)(3z+1)\le 8$ then:

$$\frac{1}{6x+1}+\frac{1}{6y+1}+\frac{1}{6z+1}\ge 1$$

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$$Let\ 3x=a,3y=b,3z=c\ .$$

$$(3x+1)(3y+1)(3z+1)\le 8\ or,(a+1)(b+1)(c+1)\le 8$$

$$2\sqrt{a}\cdot 2\sqrt{b}\cdot 2\sqrt{c}\overset{AM-GM}{\le}(a+1)(b+1)(c+1)\le 8$$

$$\sqrt{abc}\le 1\ ,abc\le 1$$

$$a+b+c\overset{AM-GM}{\ge}3\sqrt[3]{abc}\ge 3abc\ ,since\ 0<abc\le 1\ then\ \sqrt[3]{abc}\ge abc\ (1)$$

$$\sum\big((1+2a)(1+2b)\big)=\sum(1+2(a+b)+4ab)=3+4\sum a+4\sum ab=$$

$$=1+2\left(1+\sum a\right)+2\sum a+4\sum ab\overset{abc\le 1\ \&\ AM-GM}{\ge}$$

$$\ge\ 1+2\big(abc+\ 3\sqrt[3]{abc}\big)+2\sum a+4\sum ab\ge$$

$$\overset{(1)}{\ge}\ 1+2(abc+\ 3abc)+2\sum a+4\sum ab=1+8abc+2\sum a+4\sum ab\ (2)$$

$$\prod(1+2a)=1+2\sum a+4\sum ab+8abc$$

$$\frac{1}{6x+1}+\frac{1}{6y+1}+\frac{1}{6z+1}=\sum\frac{1}{6x+1}=\sum\frac{1}{2a+1}=$$

$$=\frac{\sum\big((1+2a)(1+2b)\big)}{\prod(1+2a)}\overset{(2)}{\ge}\frac{1+8abc+2\sum a+4\sum ab}{1+8abc+2\sum a+4\sum ab}=1$$

$$Equality\ holds\ for\ a=b=c=1\ or\ x=y=z=\frac{1}{3}.$$