ROMANIAN MATHEMATICAL MAGAZINE

If $x,y>0, xy=1$ and $n\in\mathbb{N}$, then :

$$\frac{x^{n+1}}{y^n+1}+\frac{y^{n+1}}{x^n+1}\geq 1$$

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We shall first prove that : $\frac{x^{n+1}}{y^n+\lambda}+\frac{y^{n+1}}{x^n+\lambda}\geq\frac{2}{\lambda+1}\ \forall\ \lambda\geq 0$

WLOG we may assume : $x\geq y$ and then : $\frac{x^n}{y^n+\lambda}\geq\frac{y^n}{x^n+\lambda}$ and so,

$$\text{LHS}\overset{\text{Chebyshev}}{\geq}\frac{x+y}{2}.\left(\frac{x^n}{y^n+\lambda}+\frac{y^n}{x^n+\lambda}\right)\overset{\text{AM-GM}}{\geq}\sqrt{xy}.\left(\frac{a^2+b^2+(a+b)\lambda}{ab+(a+b)\lambda+\lambda^2}\right)$$

$$(a=x^n,b=y^n)\overset{xy=1}{=}\frac{a^2+b^2+(a+b)\lambda}{ab+(a+b)\lambda+\lambda^2}\overset{?}{\geq}\frac{2}{\lambda+1}$$

$$\Leftrightarrow a^2+b^2+(a+b)\lambda+\lambda(a^2+b^2)+(a+b)\lambda^2\overset{?}{\geq}2ab+2(a+b)\lambda+2\lambda^2$$

$$\Leftrightarrow (a-b)^2+\lambda\left(a^2+b^2-(a+b)\right)+\lambda^2(a+b-2)\overset{?}{\geq}0\rightarrow\text{true}\ \because a^2+b^2\geq$$

$$\frac{1}{2}(a+b)^2\overset{\text{AM-GM}}{\geq}\frac{1}{2}(a+b).2\sqrt{ab}\overset{ab=1}{=}a+b\Rightarrow a^2+b^2-(a+b)\geq 0$$

and $\because a+b\overset{\text{AM-GM}}{\geq}2\sqrt{ab}\overset{ab=1}{=}2\Rightarrow a+b-2\geq 0$ and $\because\lambda\geq 0\ \therefore\frac{x^{n+1}}{y^n+\lambda}+\frac{y^{n+1}}{x^n+\lambda}$

$\overset{(*)}{\geq}\frac{2}{\lambda+1}\ \forall\ x,y>0\ \big|\ xy=1$ and $n\in\mathbb{N},\lambda\geq 0$ and putting $\lambda=1$ in $(*)$, we get :

$\frac{x^{n+1}}{y^n+1}+\frac{y^{n+1}}{x^n+1}\geq 1\ \forall\ x,y>0\ \big|\ xy=1$ and $n\in\mathbb{N}$,'' $=$'' iff $x=y=1$ (QED)