ROMANIAN MATHEMATICAL MAGAZINE

If $x,y>0, xy=1$ and $n\in\mathbb{N}, 0\le\lambda\le 1$, then :

$$\frac{x^{n+1}}{y^n+\lambda}+\frac{y^{n+1}}{x^n+\lambda}\ge\frac{2}{\lambda+1}$$

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WLOG we may assume : $x\ge y$ and then : $\frac{x^n}{y^n+\lambda}\ge\frac{y^n}{x^n+\lambda}$ and so,

$$LHS\overset{Chebyshev}{\ge}\frac{x+y}{2}.\left(\frac{x^n}{y^n+\lambda}+\frac{y^n}{x^n+\lambda}\right)\overset{AM-GM}{\ge}\sqrt{xy}.\left(\frac{a^2+b^2+(a+b)\lambda}{ab+(a+b)\lambda+\lambda^2}\right)$$

$$(a=x^n,b=y^n)\overset{xy=1}{=}\frac{a^2+b^2+(a+b)\lambda}{ab+(a+b)\lambda+\lambda^2}\overset{?}{\ge}\frac{2}{\lambda+1}$$

$$\Leftrightarrow a^2+b^2+(a+b)\lambda+\lambda(a^2+b^2)+(a+b)\lambda^2\overset{?}{\ge}2ab+2(a+b)\lambda+2\lambda^2$$

$$\Leftrightarrow (a-b)^2+\lambda\left(a^2+b^2-(a+b)\right)+\lambda^2(a+b-2)\overset{?}{\ge}0\to true\because a^2+b^2\ge$$

$$\frac{1}{2}(a+b)^2\overset{AM-GM}{\ge}\frac{1}{2}(a+b).2\sqrt{ab}\overset{ab=1}{=}a+b\Rightarrow a^2+b^2-(a+b)\ge 0$$

and $\because a+b\overset{AM-GM}{\ge}2\sqrt{ab}\overset{ab=1}{=}2\Rightarrow a+b-2\ge 0$ and $\because\lambda\ge 0\therefore\frac{x^{n+1}}{y^n+\lambda}+\frac{y^{n+1}}{x^n+\lambda}$

$$\ge\frac{2}{\lambda+1}\ \forall x,y>0\,|\,xy=1 \text{ and } n\in\mathbb{N}, 0\le\lambda\le 1,''='' \text{ iff } x=y=1\ (QED)$$