ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0,\ \lambda\geq 2, abc=1$ then:

$$\sum\frac{1}{a+\lambda}\leq\frac{3}{3+\lambda}$$

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$$\sum ab \overset{AM-GM}{\geq} 3(abc)^{\frac{2}{3}} \overset{abc=1}{=} 3\ (1)$$
$$\sum a \overset{AM-GM}{\geq} 3(abc)^{\frac{1}{3}} \overset{abc=1}{=} 3\ (2)$$

$$\sum\frac{1}{a+\lambda}=\frac{(a+\lambda)(b+\lambda)+(b+\lambda)(c+\lambda)+(c+\lambda)(a+\lambda)}{(a+\lambda)(b+\lambda)(c+\lambda)}=$$

$$=\frac{3\lambda^2+2\lambda(a+b+c)+ab+bc+ca}{\lambda^3+\lambda^2(a+b+c)+\lambda(ab+bc+ca)+abc}\overset{abc=1}{=}$$

$$=\frac{3\lambda^2+2\lambda(a+b+c)+ab+bc+ca}{\lambda^3+\lambda^2(a+b+c)+\lambda(ab+bc+ca)+1}$$

$$\textit{We need to show:}$$
$$\frac{3\lambda^2+2\lambda(a+b+c)+ab+bc+ca}{\lambda^3+\lambda^2(a+b+c)+\lambda(ab+bc+ca)+1}\leq 1$$

$$\lambda^3+\lambda^2(a+b+c)+\lambda(ab+bc+ca)+1\geq 3\lambda^2+2\lambda(a+b+c)+ab+bc+ca$$

$$\lambda^3+(\lambda^2-2\lambda)(a+b+c)+(\lambda-1)(ab+bc+ca)+1-3\lambda^2\geq 0$$
$$\lambda^3+(\lambda^2-2\lambda).3+(\lambda-1)3+1-3\lambda^2\overset{(1)\&(2)}{\geq}0$$
$$\lambda^3-3\lambda-2\geq 0$$
$$(\lambda-2)(\lambda^2+2\lambda+1)\geq 0\ true\ as\ \lambda\geq 2$$

Equality holds for a=b=c=1.