ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0$, $xyz=1$ and $n\in N$, $n\geq 2$ then:

$$\sum \frac{x^n}{x+y+1}\geq 1$$

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$$3=3.(1)^{\frac{1}{3}}\overset{xyz=1}{=}3(xyz)^{\frac{1}{3}}\overset{AM-GM}{\leq}x+y+z\ (1)$$

$$\left(\sum \frac{x^n}{x+y+1}\right)\left(\sum (x+y+1)\right)(1+1+1)^{n-2}\overset{Holder}{\geq}\frac{(x+y+z)^n}{3^{n-2}(\sum(x+y+1))}=$$

$$=\frac{(x+y+z)^n}{3^{n-2}\left(3+2(x+y+z)\right)}\overset{(1)}{\geq}\frac{(x+y+z)^n}{3^{n-2}\left((x+y+z)+2(x+y+z)\right)}=$$

$$=\frac{(x+y+z)^{n-1}}{3^{n-1}}\overset{AM-GM}{\geq}\frac{3^{n-1}(xyz)^{\frac{n-1}{3}}}{3^{n-1}}=1\ (as\ xyz=1)$$

Equality holds for $x=y=z=1$.