ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\geq 1$, then :

$$\sum_{cyc}\frac{1}{1+a}\geq\frac{9}{3+\sum_{cyc}\sqrt{ab}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\sqrt{a},y=\sqrt{b},z=\sqrt{c} \text{ and then : } \sum_{cyc}\frac{1}{1+a}\geq\frac{9}{3+\sum_{cyc}\sqrt{ab}}$$

$$\Leftrightarrow\sum_{cyc}\frac{1}{1+x^2}\geq\frac{9}{3+\sum_{cyc}xy}\Leftrightarrow\frac{3+2\sum_{cyc}x^2+\sum_{cyc}x^2y^2}{1+x^2y^2z^2+\sum_{cyc}x^2+\sum_{cyc}x^2y^2}\geq\frac{9}{3+\sum_{cyc}xy}$$

$$\Leftrightarrow 3\sum_{cyc}xy+2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2y^2\right)\overset{(*)}{\geq}$$

$$3\sum_{cyc}x^2+6\sum_{cyc}x^2y^2+9x^2y^2z^2 \text{ and assigning } x=u+1,y=v+1,z=w+1$$

$$(u,v,w\geq 0) \text{ and subsequently, expanding and re}-\text{arranging, } (*)\Leftrightarrow$$

$$\sum_{cyc}u^3v^3+uvw\left(\sum_{cyc}u^2v+\sum_{cyc}uv^2\right)-9u^2v^2w^2$$

$$+4\left(\sum_{cyc}u^3v^2+\sum_{cyc}u^2v^3+uvw\sum_{cyc}u^2-3uvw\sum_{cyc}uv\right)$$

$$+8\left(\sum_{cyc}u^3v+\sum_{cyc}uv^3-2uvw\sum_{cyc}u\right)+8\left(\sum_{cyc}u^3-3uvw\right)$$

$$+4\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)\overset{(**)}{\geq}0$$

$$\text{Now, LHS of } (**)\geq 3u^2v^2w^2+\left(\sum_{cyc}uv\right)\left(\sum_{cyc}u^2v^2-uvw\sum_{cyc}u\right)$$

$$+uvw\sum_{cyc}\left(w(u^2+v^2-2uv)\right)-3u^2v^2w^2$$

$$+4\left(\sum_{cyc}\left(w^3(u^2+v^2)\right)-2uvw\sum_{cyc}uv+uvw\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)\right)$$

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$$+8\left(\sum_{cyc}\left(uv(u^2+v^2)\right)-2uvw\sum_{cyc}u\right)$$

$$+8\left(3uvw+\left(\sum_{cyc}u\right)\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)-3uvw\right)$$

$$\geq uvw\sum_{cyc}(w(u-v)^2)+8uvw\left(\sum_{cyc}u^2-\sum_{cyc}uv\right)+16\left(\sum_{cyc}u^2v^2-uvw\sum_{cyc}u\right)$$

$$\geq 0\left(\because \sum_{cyc}u^2v^2\geq uvw\sum_{cyc}u,\sum_{cyc}u^2\geq\sum_{cyc}uv\ and\ u^2+v^2\geq 2uv\ and\ analogs\right.$$

$$\left.\forall u,v,w\in\mathbb{R}\ and\ \therefore u,v,w\geq 0\right)$$

$$\Rightarrow (**)\Rightarrow (*)\ \text{is true}\ \therefore \sum_{cyc}\frac{1}{1+a}\geq\frac{9}{3+\sum_{cyc}\sqrt{ab}}\ \forall a,b,c\geq 1,$$

$$''='' \text{ iff } a=b=c=1\ (QED)$$