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If $a,b,c>0, a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $\lambda\geq 0$, then :

$$\sum_{cyc}\sqrt{\frac{a^3}{\lambda+3bc}}\geq\frac{3}{\sqrt{\lambda+3}}$$

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$$\sum_{cyc}\sqrt{\frac{a^3}{\lambda+3bc}}=\sum_{cyc}\frac{a^2}{\sqrt{\lambda a+3abc}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\sqrt{\lambda a+3abc}}\overset{\text{CBS}}{\geq}$$

$$\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{3\lambda\sum_{cyc}a+27abc}}\overset{?}{\geq}\frac{3}{\sqrt{\lambda+3}}$$

$$\Leftrightarrow\lambda\left(\sum_{cyc}a\right)^4+3\left(\sum_{cyc}a\right)^4\overset{?}{\underset{(*)}{\geq}}27\lambda\left(\sum_{cyc}a\right)+243abc$$

$$\text{Now, }\sum_{cyc}ab=abc\sum_{cyc}a\leq\frac{1}{3}\left(\sum_{cyc}ab\right)^2\Rightarrow\sum_{cyc}ab\geq 3\Rightarrow\left(\sum_{cyc}a\right)^2\geq 3\sum_{cyc}ab\geq 9$$

$$\Rightarrow\sum_{cyc}a\geq 3 \text{ and so, since } \lambda\geq 0,\lambda\left(\sum_{cyc}a\right)^4+3\left(\sum_{cyc}a\right)^4\geq$$

$$\lambda\left(\sum_{cyc}a\right).27+9\left(\sum_{cyc}a\right)^3\overset{A-G}{\geq}27\lambda\left(\sum_{cyc}a\right)+243abc\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\sqrt{\frac{a^3}{\lambda+3bc}}\geq\frac{3}{\sqrt{\lambda+3}}\ \forall a,b,c>0\ \Big|\ a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$