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If $a,b,c>0$ and $\lambda\le 6$, then :

$$\sum_{cyc} a^3+\lambda abc\ge(\lambda+3)\frac{\sum_{cyc}a^2}{\sum_{cyc}\frac{1}{a}}$$

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Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $=s,R,r$ (say); & then : $\sum_{cyc} a=s\to(1), abc=r^2s\to(2), \sum_{cyc} ab=4Rr+r^2\to(3)$,

$$\sum_{cyc} a^2=s^2-8Rr-2r^2\to(4)\ \&\ \sum_{cyc} a^3=s(s^2-12Rr)\to(5)\ and$$

$$\sum_{cyc} a^3+\lambda abc-(\lambda+3)\frac{\sum_{cyc}a^2}{\sum_{cyc}\frac{1}{a}}$$

$$=\sum_{cyc} a^3-3abc\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}-\frac{\lambda abc}{\sum_{cyc}ab}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$\overset{\lambda\le 6}{\ge}\sum_{cyc} a^3-3abc\frac{\sum_{cyc}a^2}{\sum_{cyc}ab}-\frac{6abc}{\sum_{cyc}ab}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\overset{via\ (2),(3),(4)\ and\ (5)}{=}$$

$$\frac{s(s^2-12Rr)(4Rr+r^2)-3r^2s(s^2-8Rr-2r^2)-6r^2s(s^2-12Rr-3r^2)}{4Rr+r^2}$$

$$=\frac{4s\left((R-2r)s^2-r(12R^2-21Rr-6r^2)\right)}{4R+r}$$

$$\overset{Gerretsen}{\ge}\frac{4s\left((R-2r)(16Rr-5r^2)-r(12R^2-21Rr-6r^2)\right)}{4R+r}=\frac{16rs(R-2r)^2}{4R+r}$$

$$\overset{Euler}{\ge}0\ \therefore\sum_{cyc} a^3+\lambda abc\ge(\lambda+3)\frac{\sum_{cyc}a^2}{\sum_{cyc}\frac{1}{a}}\ \forall\ a,b,c>0\wedge\lambda\le 6,$$

$''='' \text{ iff } a=b=c\ (QED)$