ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ then:

$$\sqrt[3]{ln(1+e^a)ln(1+e^b)ln(1+e^c)} \leq ln\left(1+\sqrt[3]{e^{a+b+c}}\right)$$

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Solution by Christos Tsifakis-Greece

Let be $f(x) = ln\big(ln(1+e^x)\big), x \in \mathbb{R}$.

$$f'(x) = \frac{e^x}{(1+e^x)ln(1+e^x)}$$

It is known that:

$$ln(1+t) < t, t > -1 \Rightarrow ln(1+e^x) < 1 + e^x - 1 = e^x$$
$$ln(1+e^x) - e^x < 0$$

$$f''(x) = \frac{e^x(ln(1+e^x) - e^x)}{(1+e^x)^2 ln^2(1+e^x)} < 0 \Rightarrow$$

f −concave. By Jensen's inequality:

$$\frac{f(a)+f(b)+f(c)}{3} \leq f\left(\frac{a+b+c}{3}\right)$$

$$\frac{1}{3}\left(ln\big(ln(1+e^a)\big) + ln\left(ln\big(1+e^b\big)\right) + ln\big(ln(1+e^c)\big)\right) \leq ln\left(ln\left(1+e^{\frac{a+b+c}{3}}\right)\right)$$

$$ln\sqrt[3]{ln(1+e^a)+ln(1+e^b)+ln(1+e^c)} \leq ln\left(ln\left(1+\sqrt[3]{e^{a+b+c}}\right)\right)$$

$$\sqrt[3]{ln(1+e^a)+ln(1+e^b)+ln(1+e^c)} \leq ln\left(1+\sqrt[3]{e^{a+b+c}}\right)$$

$$\sqrt[3]{ln\big((1+e^a)(1+e^b)(1+e^c)\big)} \leq ln\left(1+\sqrt[3]{e^{a+b+c}}\right)$$

Equality holds for $a = b = c$.