RMM - Inequalities Marathon 1301 - 1400

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
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Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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1301. If $a,b,c>0$, then :

$$\frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\geq\frac{4a}{a+1}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Şerban George Florin-Romania

$$a>0, f:[0,\infty)\to\mathbb{R}, f(x)=\frac{ax+2a+x}{a+2x+1}$$

$$f'(x)=\frac{(a+1)(a+2x+1)-2(ax+2a+x)}{(a+2x+1)^2}$$

$$f'(x)=\frac{a^2+2ax+a+a+2x+1-2ax-4a-2x}{(a+2x+1)^2}$$

$$f'(x)=\frac{a^2-2a+1}{(a+2x+1)^2}=\frac{(a-1)^2}{(a+2x+1)^2}=\left(\frac{a-1}{a+2x+1}\right)^2>0$$

$$\Rightarrow f\uparrow[0,\infty)$$

$$\left.\begin{aligned}b>0\Rightarrow f(b)>f(0)=\frac{2a}{a+1}\\ c>0\Rightarrow f(c)>f(0)=\frac{2a}{a+1}\end{aligned}\right\}\Rightarrow f(b)+f(c)>\frac{2a}{a+1}+\frac{2a}{a+1}=\frac{4a}{a+1}$$

$$\Rightarrow\frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\geq\frac{4a}{a+1}$$

$$(\forall)a,b,c>0$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\geq\frac{4a}{a+1}\Leftrightarrow\frac{ab+2a+b}{a+2b+1}-1+\frac{ac+2a+c}{a+2c+1}-1$$

$$\geq\frac{4a}{a+1}-2\Leftrightarrow\frac{ab+a-b-1}{a+2b+1}+\frac{ac+a-c-1}{a+2c+1}\geq\frac{2a-2}{a+1}$$

$$\Leftrightarrow\frac{a(b+1)-(b+1)}{a+2b+1}+\frac{a(c+1)-(c+1)}{a+2c+1}\geq\frac{2(a-1)}{a+1}$$

$$\Leftrightarrow\frac{(a-1)(b+1)}{a+2b+1}+\frac{(a-1)(c+1)}{a+2c+1}\geq\frac{2(a-1)}{a+1}$$

$$\Leftrightarrow (a-1)\left(\left(\frac{b+1}{a+2b+1}-\frac{1}{a+1}\right)+\left(\frac{c+1}{a+2c+1}-\frac{1}{a+1}\right)\right)\geq 0$$

$$\Leftrightarrow \frac{(a-1)}{a+1}\left(\frac{ab+a+b+1-a-2b-1}{a+2b+1}+\frac{ac+a+c+1-a-2c-1}{a+2c+1}\right)\geq 0$$

$$\Leftrightarrow \frac{(a-1)}{a+1}\left(\frac{b(a-1)}{a+2b+1}+\frac{c(a-1)}{a+2c+1}\right)\geq 0$$

$$\Leftrightarrow \frac{(a-1)^2}{a+1}\left(\frac{b}{a+2b+1}+\frac{c}{a+2c+1}\right)\geq 0\rightarrow \text{true}$$

$$\therefore \frac{ab+2a+b}{a+2b+1}+\frac{ac+2a+c}{a+2c+1}\geq\frac{4a}{a+1}\ \forall\, a,b,c>0,''='' \text{ iff } a=1\ (\text{QED})$$

1302. If $a_k>0\ (k=1,2,\dots,n)$ and $s=\sum_{k=1}^{n}a_k$, then :

$$\sum_{k=1}^{n}a_k^3\geq\frac{1}{n(n-1)}\left(\sum_{k=1}^{n}a_k\cdot\sqrt{s-a_k}\right)^2$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{n(n-1)}\left(\sum_{k=1}^{n}a_k\cdot\sqrt{s-a_k}\right)^2\overset{CBS}{\leq}\frac{1}{n(n-1)}\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}(s-a_k)\right)$$

$$=\frac{1}{n(n-1)}\left(\sum_{k=1}^{n}a_k^2\right)\left(n\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}a_k\right)=\frac{n-1}{n(n-1)}\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}a_k\right)$$

$$\Rightarrow\frac{1}{n(n-1)}\left(\sum_{k=1}^{n}a_k\cdot\sqrt{s-a_k}\right)^2\leq\frac{1}{n}\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}a_k\right)\rightarrow(1)$$

$$\text{Now, }\sum_{k=1}^{n}a_k^3\overset{Chebyshev}{\geq}\frac{1}{n}\left(\sum_{k=1}^{n}a_k^2\right)\left(\sum_{k=1}^{n}a_k\right)\overset{via\ (1)}{\geq}\frac{1}{n(n-1)}\left(\sum_{k=1}^{n}a_k\cdot\sqrt{s-a_k}\right)^2,$$

$$''='' \text{ iff } a_1=a_2=\cdots=a_n\ (\text{QED})$$

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1303. If $x_k > 0\ (k = 1, 2, \dots, n)$, then :

$$\frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{3}{x_1x_2+x_2x_3+x_3x_1}}-\frac{n}{\sum_{k=1}^{n}x_k}\geq 1$$

Proposed by Neculai Stanciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{3}{x_1x_2+x_2x_3+x_3x_1}}-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$\geq\frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{9}{(x_1+x_2+x_3)^2}}-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$=\frac{1}{n}\sum_{cyc}\sqrt{\frac{(x_1+x_2+x_3)^2+6(x_1+x_2+x_3)+9}{(x_1+x_2+x_3)^2}}-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$=\frac{1}{n}\sum_{cyc}\sqrt{\left(\frac{x_1+x_2+x_3+3}{x_1+x_2+x_3}\right)^2}-\frac{n}{\sum_{k=1}^{n}x_k}=\frac{1}{n}\sum_{cyc}\left(1+\frac{3}{x_1+x_2+x_3}\right)-\frac{n}{\sum_{k=1}^{n}x_k}$$

$$=\frac{1}{n}.n+\frac{3}{n}\sum_{cyc}\frac{1}{x_1+x_2+x_3}-\frac{n}{\sum_{k=1}^{n}x_k}\overset{\text{Bergstrom}}{\geq}1+\frac{3}{n}.\frac{n^2}{3\sum_{k=1}^{n}x_k}-\frac{n}{\sum_{k=1}^{n}x_k}=1$$

$$\therefore\frac{1}{n}\sum_{cyc}\sqrt{1+\frac{6}{x_1+x_2+x_3}+\frac{3}{x_1x_2+x_2x_3+x_3x_1}}-\frac{n}{\sum_{k=1}^{n}x_k}\geq 1$$

$\forall\, x_k > 0\ (k = 1, 2, \dots, n)$, $''='' $ iff $x_1 = x_2 = \cdots = x_n$ (QED)

1304. If $a_k > 0 (k = 1, 2, \dots, n)$ and $s = \sum_{k=1}^{n} a_k$, then:

$$\left(\frac{n-1}{n}\right)^2\sum_{k=1}^{n}a_k\cdot\sum_{k=1}^{n}\frac{a_k}{(s-a_k)^2}\geq 1$$

Proposed by Neculai Stanciu – Romania

Solution by Marian Ursărescu – Romania

$$\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{a_k}{(s-a_k)^2} \geq \frac{n^2}{(n-1)^2} \Leftrightarrow \sum_{k=1}^{n} \frac{a_k^2}{(s-a_k)^2} + \sum_{k=1}^{n} \frac{a_k(s-a_k)}{(s-a_k)^2} \geq \frac{n^2}{(n-1)^2} \Leftrightarrow$$

$$\sum_{k=1}^{n} \left(\frac{a_k}{s-a_k}\right)^2 + \sum_{k=1}^{n} \frac{a_k}{s-a_k} \geq \frac{n^2}{(n-1)^2} \quad (1)$$

$$\text{But } \sum_{k=1}^{n} \frac{a_k}{s-a_k} \geq \frac{n}{n-1} \quad (2)$$

$$\text{From Cauchy } n \cdot \sum_{k=1}^{n} \left(\frac{a_k}{s-a_k}\right)^2 \geq \left(\sum_{k=1}^{n} \frac{a_k}{s-a_k}\right)^2$$

$$\sum_{k=1}^{n} \left(\frac{a_k}{s-a_k}\right)^2 \geq \frac{1}{n}\left(\sum_{k=1}^{n} \frac{a_k}{s-a_k}\right)^2 \overset{(2)}{\geq} \frac{1}{n} \cdot \frac{n^2}{(n-1)^2} = \frac{n}{(n-1)^2} \quad (3)$$

$$\text{From (1) + (2) + (3) :}$$

$$\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{a_k}{(s-a_k)^2} \geq \frac{n}{n-1} + \frac{n}{(n+1)^2} = \frac{n^2-n+n}{(n-1)^2} = \frac{n^2}{(n-1)^2}$$

1305. If $x, y, z > 0$ then:

$$3\sum_{cyc} \frac{|\cosh x - \cosh y|}{\sqrt{\sinh x \cdot \sinh y}} \geq \left(\sum_{cyc} \sqrt{|x-y|}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Ahmed Salem-Tunisia

If $g(x)$ is a convex function on D, then $\forall a, b \in D: g(a) \geq g(b) + g'(b)(a-b)$

The hyperbolic $\sinh$ function is convex on $\mathbb{R}_{>0}$: $\sinh(a) \geq \sinh(b) + \cosh(b)\,(a-b)$.

W.L.O.G consider $x > y > 0$.

$$\cosh x - \cosh y = \int_y^x \sinh t\, dt \geq \int_y^x \left(\sinh\left(\frac{x+y}{2}\right) + \cosh\left(\frac{x+y}{2}\right)\left(t - \frac{x+y}{2}\right)\right) dt =$$

$$= \sinh\left(\frac{x+y}{2}\right)(x-y)$$

$$\sinh^2\left(\frac{x+y}{2}\right) - \underbrace{\sinh x \sinh y}_{=\frac{1}{2}(\cosh(x+y)-\cosh(x-y))} =$$

$$Also, \cosh\alpha = 2\cosh^2\left(\frac{\alpha}{2}\right) - 1$$

$$= \sinh^2\left(\frac{x+y}{2}\right) - \left(\cosh^2\left(\frac{x+y}{2}\right) - \cosh^2\left(\frac{x-y}{2}\right)\right)$$

$$= \cosh^2\left(\frac{x-y}{2}\right) - 1 = \sinh^2\left(\frac{x-y}{2}\right) \geq 0 \Rightarrow \sinh\left(\frac{x+y}{2}\right) \geq \sqrt{\sinh x \sinh y}$$

Therefore, $\forall x, y \in \mathbb{R}_{>0}: \frac{|\cosh x - \cosh y|}{\sqrt{\sinh x \sinh y}} \geq |x-y|$

$$3\sum_{cyc}\frac{|\cosh x - \cosh y|}{\sqrt{\sinh x \sinh y}} \geq 3\sum_{cyc}|x-y| \overset{Cauchy-Schwarz}{\geq} \left(\sum_{cyc}\sqrt{|x-y|}\right)^2$$

The inequality is not always true if the power is changed from 2 to $\frac{1}{2}$.

1306. If $a, b, c \in [0, 2]$ then:

$$\frac{a^2+b^2+c^2}{3} \leq \frac{(a+b+c)^2}{3} + 1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sakthi Vel-India

$$\frac{(a+b+c)^2}{3} + 1 - \frac{(a^2+b^2+c^2)}{3} \geq 0$$

$$\frac{a^2+b^2+c^2}{3} + \frac{2ab+2bc+2ac}{3} - \frac{(a^2+b^2+c^2)}{3} + 1 \geq$$

$$\geq \frac{6[abc]^{\frac{3}{6}}}{3} + 1 = 2(abc)^{\frac{1}{2}} + 1 \geq 0$$

which is true since $a, b, c \in [0, 2]$

Solution 2 by Fayssal Abdelli-Algeria

$$a, b, c \in [0, 2]$$

$$\frac{a^2+b^2+c^2}{3} \overset{?}{\leq} \frac{(a+b+c)^2}{3} + 1$$

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ac)$$

$$\Leftrightarrow (a+b+c)^2 \geq a^2+b^2+c^2 \text{ because } 2(ab+bc+ca) \geq 0$$

$$\Rightarrow \frac{(a+b+c)^2}{3} \geq \frac{(a^2+b^2+c^2)}{3} \Rightarrow 1 + \frac{(a+b+c)^2}{3} \geq \frac{a^2+b^2+c^2}{3}$$

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1307. If $0 < a, b, c \leq 1$ then:

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9 + 2(a^2+b^2+c^2-ab-bc-ca)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Khaled Abd Imouti-Syria

$$e_1 = 3 + \left(\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}\right) - 2(a^2+b^2+c^2) + 2(ab+bc+ca) \overset{?}{\geq} 9$$

$$\left(\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}\right) - 2(a^2+b^2+c^2) + 2(ab+bc+ca) \overset{?}{\geq} 6$$

$$\left.\begin{array}{c} by\ AM-GM: \frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b} \geq 6 \\ by\ Cauchy\ Schwarz: 2(a^2+b^2+c^2) - 2(ab+bc+ca) \geq 0 \end{array}\right\} \Rightarrow$$

$$\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{a} - 2(a^2+b^2+c^2) + 2(ab+bc+ca) \geq 6$$

Equality occurred when $a = b = c = 1$.

Solution 2 by Ravi Prakash-New Delhi-India

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - 9 - 2(a^2+b^2+c^2-ab-bc-ca)$$

$$= \left(\frac{b}{a}+\frac{a}{b}-2\right) + \left(\frac{c}{a}+\frac{a}{c}-2\right) + \left(\frac{b}{c}+\frac{c}{b}-2\right) - [(a-b)^2+(b-c)^2+(c-a)^2]$$

$$= (a-b)^2\left(\frac{1}{ab}-1\right) + (b-c)^2\left(\frac{1}{bc}-1\right) + (c-a)^2\left(\frac{1}{ca}-1\right)$$

$$= \frac{(a-b)^2(1-ab)}{ab} + \frac{(b-c)^2(1-bc)}{bc} + \frac{(c-a)^2(1-ca)}{ca} \geq 0 \quad [\because 0 < a, b, c \leq 1]$$

Equality when $a = b = c = 1$.

1308. If $a, b, c > 0, and\ a+b+c = 1, \lambda \geq \frac{7}{9}$ then:

$$\frac{1}{\lambda+a^2} + \frac{1}{\lambda+b^2} + \frac{1}{\lambda+c^2} \leq \frac{27}{9\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$\boldsymbol{Lemma}$: $\boldsymbol{For}\ x\in\left(0,\frac{1}{3}\right)\boldsymbol{and}\ \lambda\geq\frac{7}{9}$:

$$\frac{1}{\lambda+x^2}\leq\frac{9}{9\lambda+1}+\frac{18-54x}{(9\lambda+1)^2}$$

$\boldsymbol{Proof}$:

$$\frac{1}{\lambda+x^2}\leq\frac{9}{9\lambda+1}+\frac{18-54x}{(9\lambda+1)^2}\ \boldsymbol{or},$$

$$(9\lambda+1)^2\leq(27+81\lambda-54x)(\lambda+x^2)\boldsymbol{or},$$

$$54x^3-x^2(27+81\lambda)+54x\lambda-9\lambda+1\leq 0\ \boldsymbol{or},$$

$$2p^3-p^2(3+9\lambda)+18p\lambda-9\lambda+1\overset{p=3x>0}{\leq}0\ \boldsymbol{or}$$

$$(p-1)^2(2p+1-9\lambda)\leq 0\ \boldsymbol{True}\ .$$

$$\boldsymbol{Since}\ x<\frac{1}{3}\ \boldsymbol{so}\ p=3x<1\ and\ 2p+1<3\ and$$
$$9\lambda\geq 9.\frac{7}{9}=7\ \boldsymbol{clearly}\ (2p+1-9\lambda)<0$$

$$\frac{1}{\lambda+a^2}+\frac{1}{\lambda+b^2}+\frac{1}{\lambda+c^2}\overset{lemma}{\leq}\sum\left[\frac{9}{9\lambda+1}+\frac{18-54a}{(9\lambda+1)^2}\right]=$$

$$=\frac{27}{9\lambda+1}+\frac{54-54\sum a}{(9\lambda+1)^2}=\frac{27}{9\lambda+1},\boldsymbol{as}\ \sum a=1$$

$$\boldsymbol{Equality\ for}:\ a=b=c=\frac{1}{3}$$

1309. If $a,b,c>0\ ,a+b+c=3,\lambda\geq 0$ **then:**

$$\frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}\geq\frac{3}{\lambda+2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{a^3}{a^2+\lambda ab+b^2}=a-\frac{a(\lambda ab+b^2)}{a^2+\lambda\, ab+b^2}\overset{AM-GM}{\geq}a-\frac{ab(\lambda a+b)}{2ab+\lambda ab}=a-\frac{\lambda a+b}{\lambda+2}$$

$$\frac{a^3}{a^2+\lambda ab+b^2}+\frac{b^3}{b^2+\lambda bc+c^2}+\frac{c^3}{c^2+\lambda ca+a^2}\geq$$

$$\geq\sum\left[a-\frac{\lambda a+b}{\lambda+2}\right]=\sum a-\frac{(\lambda+1)(a+b+c)}{\lambda+2}=$$

$$=3-\frac{3\lambda+3}{\lambda+2}\left(as\ \sum a=3\right)=\frac{3}{\lambda+2}$$

$$Equality\ for\ \ a=b=c=1.$$

1310. If $x,y,z>0,\ xy+yz+zx=3\ ,\ \lambda\geq 2$ **then:**

$$\frac{1}{x^2+\lambda}+\frac{1}{y^2+\lambda}+\frac{1}{z^2+\lambda}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$x^2+y^2+z^2\geq\ xy+yz+zx=3$$

$$(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)=\sum x^2+6$$

$$\sum\frac{1}{x^2+\lambda}=\frac{1}{\lambda}\sum\left(1-\frac{x^2}{x^2+\lambda}\right)=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\frac{x^2}{x^2+\lambda}\overset{CBS}{\leq}$$

$$\leq\frac{3}{\lambda}-\frac{1}{\lambda}\left(\frac{(x+y+z)^2}{x^2+y^2+z^2+3\lambda}\right)=$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\left(\frac{x^2+y^2+z^2+6}{x^2+y^2+z^2+3\lambda}\right)\overset{p=\sum x^2}{=}\frac{3}{\lambda}-\frac{1}{\lambda}\left(\frac{p+6}{p+3\lambda}\right)=$$

$$\frac{2}{\lambda}+\frac{1}{\lambda}\left(1-\frac{p+6}{p+3\lambda}\right)=\frac{2}{\lambda}+\frac{1}{\lambda}\left(\frac{3\lambda-6}{p+3\lambda}\right)\overset{p=\sum x^2\geq 3}{\leq}$$

$$\leq\frac{2}{\lambda}+\frac{1}{\lambda}\left(\frac{3\lambda-6}{3+3\lambda}\right)=\frac{1}{\lambda}\frac{9\lambda}{3+3\lambda}=\frac{3}{\lambda+1}$$

$$Equality\ for\ a=b=c=1$$

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1311. If $a,b,c>0, a^2+b^2+c^2=3$, $\lambda \geq 2+\sqrt{3}$ then:

$$\sum_{cyc} \frac{1}{\lambda-a} \leq \frac{3}{\lambda-1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ p=a^2, q=b^2, r=c^2\ and\ a^2+b^2+c^2=3\ or\ p+q+r=3$$

$Lemma$:

$$For\ x\in(0,3) and\ \lambda \geq 2+\sqrt{3}, \frac{1}{\lambda-\sqrt{x}} \leq \frac{2\lambda-3+x}{2(\lambda-1)^2}$$

$Proof$:

$$\frac{1}{\lambda-\sqrt{x}} \leq \frac{2\lambda-3+x}{2(\lambda-1)^2}\ or,$$
$$2(\lambda-1)^2 \leq [2\lambda-3+x](\lambda-\sqrt{x}) or,$$
$$\lambda-2+x\lambda-\sqrt{x}(2\lambda-3)-x^{\frac{3}{2}} \geq 0\ or$$
$$u^3-u^2\lambda+u(2\lambda-3)-\lambda+2 \overset{x=u^2}{\leq} 0\ or,$$

$$(u-1)(u+2-\lambda) \leq 0$$

$$True, since\ x\in(0,3) and\ u=\sqrt{x}<\sqrt{3}, u+2<2+\sqrt{3}$$
$$and\ \lambda \geq 2+\sqrt{3}, so\ (u+2-\lambda)<0$$

$$Now \sum \frac{1}{\lambda-a}=\sum \frac{1}{\lambda-\sqrt{a^2}}=\sum \frac{1}{\lambda-\sqrt{p}} \overset{Lemma}{\leq}$$
$$\sum \frac{2\lambda-3+p}{2(\lambda-1)^2}=\frac{3(2\lambda-3)+\sum p}{2(\lambda-1)^2}=$$

$$=\frac{6\lambda-9+3}{2(\lambda-1)^2}\left(as \sum p=3\right)=\frac{3}{\lambda-1}$$

$$Equality\ for\ a=b=c=1$$

1312. If $a,b,c>0, abc=1, \lambda \geq 2$ then:

$$\sum_{cyc} \frac{1}{a^2+\lambda} \leq \frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum a^2 \overset{AM-GM}{\geq} 3(abc)^{\frac{2}{3}} = 3 \qquad (1),$$

$$since\ abc = 1\ and$$

$$(a+b+c)^2 = \sum a^2 + 2\sum ab \overset{AM-GM}{\geq} \sum a^2 + 6(abc)^{\frac{2}{3}} = \sum a^2 + 6 \qquad (2)$$

$$\sum \frac{1}{a^2+\lambda} = \frac{1}{\lambda}\sum\left(1-\frac{a^2}{a^2+\lambda}\right) = \frac{1}{\lambda}\left[3-\sum\frac{a^2}{a^2+\lambda}\right] \overset{CBS}{\leq}$$

$$\leq \frac{1}{\lambda}\left[3-\frac{(a+b+c)^2}{\sum a^2+3\lambda}\right] \overset{(2)}{\leq} \frac{1}{\lambda}\left[3-\frac{\sum a^2+6}{\sum a^2+3\lambda}\right] =$$

$$\leq \frac{1}{\lambda}\left[2+1-\frac{\sum a^2+6}{\sum a^2+3\lambda}\right] = \frac{1}{\lambda}\left(2+\frac{3\lambda-6}{\sum a^2+3\lambda}\right) \overset{(1)}{\leq}$$

$$\leq \frac{1}{\lambda}\left(2+\frac{3\lambda-6}{3+3\lambda}\right) = \frac{1}{\lambda}\frac{9\lambda}{(3+3\lambda)} = \frac{3}{\lambda+1}$$

$$Equality\ for\ a=b=c=1$$

1313. If $a,b,c>0,\lambda\geq 2$ then:

$$\sum_{cyc}\frac{a}{\sqrt{\lambda a+b}} \leq \sqrt{\frac{3(a+b+c)}{\lambda+1}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{a}{\lambda a+b} = \frac{1}{\lambda}\sum\frac{a\lambda}{\lambda a+b} = \frac{1}{\lambda}\sum\left(1-\frac{b}{\lambda a+b}\right) = \frac{3}{\lambda}-\frac{1}{\lambda}\sum\frac{b^2}{\lambda ab+b^2} \overset{CBS}{\leq}$$

$$\leq \frac{3}{\lambda}-\frac{1}{\lambda}\frac{(a+b+c)^2}{(\sum\lambda ab)+\sum b^2} = \frac{3}{\lambda}-\frac{1}{\lambda}\frac{\sum a^2+2\sum ab}{(\sum\lambda ab)+\sum b^2} =$$

$$= \frac{2}{\lambda}+\frac{1}{\lambda}-\frac{1}{\lambda}\frac{\sum a^2+2\sum ab}{(\sum\lambda ab)+\sum b^2} = \frac{2}{\lambda}+\frac{1}{\lambda}\frac{\left((\lambda-2)\sum ab\right)}{\sum\lambda ab+\sum b^2} \overset{\sum a^2\geq\sum ab}{\leq}$$

$$\leq \frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{\left((\lambda-2)\sum ab\right)}{\sum \lambda ab+\sum ab}=\frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{\left((\lambda-2)\sum ab\right)}{(\lambda+1)\sum ab}=\frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{(\lambda-2)}{(\lambda+1)}=\frac{3}{\lambda+1}\quad(1)$$

$$\sum\frac{a}{\sqrt{\lambda a+b}}=\sum\sqrt{a}.\sqrt{\frac{a}{\lambda a+b}}\overset{CBS}{\leq}\sqrt{(a+b+c)\sum\frac{a}{\lambda a+b}}\overset{(1)}{\leq}\sqrt{\frac{3(a+b+c)}{\lambda+1}}$$

$$Equality\ for\ a=b=c=1$$

1314. If $a,b,c>0,\lambda\geq 2$ then:

$$\sum_{cyc}\frac{a}{\lambda a+b}\leq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{a}{\lambda a+b}=\frac{1}{\lambda}\sum\frac{a\lambda}{\lambda a+b}=\frac{1}{\lambda}\sum\left(1-\frac{b}{\lambda a+b}\right)=$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\sum\frac{b^2}{\lambda ab+b^2}\overset{CBS}{\leq}\frac{3}{\lambda}-\frac{1}{\lambda}\frac{(a+b+c)^2}{(\sum\lambda ab)+\sum b^2}=$$

$$=\frac{3}{\lambda}-\frac{1}{\lambda}\frac{\sum a^2+2\sum ab}{(\sum\lambda ab)+\sum b^2}=\frac{2}{\lambda}+\frac{1}{\lambda}-\frac{1}{\lambda}\frac{\sum a^2+2\sum ab}{(\sum\lambda ab)+\sum b^2}=$$

$$=\frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{\left((\lambda-2)\sum ab\right)}{\sum\lambda ab+\sum b^2}\overset{\sum a^2\geq\sum ab}{\leq}\frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{\left((\lambda-2)\sum ab\right)}{\sum\lambda ab+\sum ab}=$$

$$=\frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{\left((\lambda-2)\sum ab\right)}{(\lambda+1)\sum ab}=\frac{2}{\lambda}+\frac{1}{\lambda}\cdot\frac{(\lambda-2)}{(\lambda+1)}=\frac{3}{\lambda+1}$$

$$Equality\ for\ a=b=c=1$$

1315. If $a,b,c>0,abc\geq 1,\lambda\geq 0$ then:

$$\sum_{cyc}\frac{a^2}{bc+\lambda}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$ab+bc+ca \overset{AM-GM}{\geq} 3(abc)^{\frac{2}{3}} \geq 3(since\ abc \geq 1) \quad (1)$$

$$\sum \frac{a^2}{bc+\lambda} \overset{Bergstrom}{\geq} \frac{(a+b+c)^2}{(\sum ab)+3\lambda} \overset{(\sum ab)\leq \frac{(\sum a)^2}{3}}{\geq}$$

$$\geq \frac{3\sum ab}{\sum ab+3\lambda} \overset{(1)}{\geq} \frac{3\sum ab}{\sum ab+\lambda\sum ab} = \frac{3}{\lambda+1}$$

$$Equality\ holds\ for\ a=b=c=1$$

1316. If $a,b,c>0$ then:

$$\left(\sum_{cyc} \frac{a}{\sqrt[3]{(a+2b)}}\right)^{\frac{3}{2}} \geq \sum_{cyc} a$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\left(\sum \frac{a}{\sqrt[3]{(a+2b)}}\right)^{\frac{3}{2}} = \left(\sum \frac{a^{\frac{4}{3}}}{\sqrt[3]{(a^2+2ab)}}\right)^{\frac{3}{2}} \overset{Radon}{\geq}$$

$$\geq \left(\frac{(a+b+c)^{\frac{4}{3}}}{(\sum a^2+2\sum ab)^{\frac{1}{3}}}\right)^{\frac{3}{2}} = \left(\frac{(\sum a)^{\frac{4}{3}}}{(\sum a)^{\frac{2}{3}}}\right)^{\frac{3}{2}} = \left(\sum a\right)^{\frac{2}{3}\cdot\frac{3}{2}} = \sum a$$

Equality holds for $a=b=c$.

1317. If $a,b,c>0$ then:

$$3\sum_{cyc} ab \cdot \sum_{cyc} \frac{a}{b} - 7\sum_{cyc} ab \geq 2\sum_{cyc} a^2$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$\sum \frac{a}{b} = \sum \frac{a^2}{ab} \overset{Bergstrom}{\geq} \frac{(\sum a)^2}{\sum ab} \quad (1)$$

$$3\sum ab\sum\frac{a}{b}-7\sum ab\overset{(1)}{\geq}3\sum ab\,\frac{(\sum a)^2}{\sum ab}-7\sum ab=$$

$$=3\sum a^2+6\sum ab\,-7\sum ab=3\sum a^2-\sum ab=$$

$$2\sum a^2+\sum a^2-\sum ab\geq 2\sum a^2\left(since\ \sum a^2\geq\sum ab\right)$$

$$Equality\ holds\ for\ a=b=c.$$

1318. If $a,b,c>0$ then:

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{16}{a+b+6}\leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Khaled Abd Imouti-Syria

$$\frac{a}{a+1}+\frac{b}{b+1}+\frac{16}{a+b+6}\leq 3\Leftrightarrow$$

$$\Leftrightarrow\frac{a}{a+1}-1+\frac{b}{b+1}-1+\frac{16}{a+b+6}-1\leq 0\Leftrightarrow$$

$$\Leftrightarrow\frac{-1}{a+1}+\frac{-1}{b+1}+\frac{10-a-b}{a+b+6}\leq 0\Leftrightarrow$$

$$\Leftrightarrow\frac{1}{a+1}+\frac{1}{b+1}\geq\frac{10-(a+b)}{(a+b)+6}\ (to\ prove)$$

$$\frac{1}{a+1}+\frac{1}{b+1}\overset{BERGSTROM}{\geq}\frac{4}{a+b+2}$$

$$Denote\ a+b=x.\ \ Remains\ to\ prove:$$

$$\frac{4}{x+2}\geq\frac{10-x}{x+6}$$

$$4x+24\geq 10x-x^2+20-2x$$

$$x^2-4x+4\geq 0$$

$$(x-2)^2\geq 0$$

Equality holds for: $a=b=1$.

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1319. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$a^3+b^3+c^3+\sqrt{5}abc\geq 3+\sqrt{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y\Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say);

$$\text{so } 2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\to(1)\Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s\to(2)\text{and such substitutions}\Rightarrow\sum_{cyc}a^3=$$

$$\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)=s^3-3.4Rrs\Rightarrow\sum_{cyc}a^3=s^3-12Rrs\to(3)$$

$$\text{Now, } a^3+b^3+c^3+\sqrt{5}abc\geq 3+\sqrt{5}\Leftrightarrow\sum_{cyc}a^3-3\geq\sqrt{5}(1-abc)$$

$$\overset{a+b+c=3}{\Leftrightarrow}\sum_{cyc}a^3-\frac{\left(\sum_{cyc}a\right)^3}{9}\geq\sqrt{5}\left(\frac{\left(\sum_{cyc}a\right)^3}{27}-abc\right)$$

$$\Leftrightarrow 3\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right)\overset{(*)}{\geq}\sqrt{5}\left(\left(\sum_{cyc}a\right)^3-27abc\right)\text{ and }\because\sqrt{5}<\frac{9}{4}\text{ and}$$

$$\left(\sum_{cyc}a\right)^3-27abc\overset{A-G}{\geq}0\therefore\text{in order to prove }(*),\text{it suffices to prove :}$$

$$3\left(9\sum_{cyc}a^3-\left(\sum_{cyc}a\right)^3\right)\geq\frac{9}{4}\left(\left(\sum_{cyc}a\right)^3-27abc\right)\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow}$$

$$4(9(s^3-12Rrs)-s^3)\geq 3(s^3-27r^2s)\Leftrightarrow 29s^2\overset{(**)}{\geq}432Rr-81r^2$$

$$\text{Again, } 29s^2\overset{\text{Gerretsen}}{\geq}29(16Rr-5r^2)\overset{?}{\geq}432Rr-81r^2\Leftrightarrow 32Rr\overset{?}{\geq}64r^2\to\text{true}$$

$$\text{via Euler}\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore a^3+b^3+c^3+\sqrt{5}abc\geq 3+\sqrt{5}$$

$$\forall a,b,c>0\,|\,a+b+c=3,''='' \text{ iff } a=b=c=1\text{ (QED)}$$

1320. If $a,b>0,\ a+b=ab$ then:

$$\frac{1}{a^2}+\frac{1}{b^2}\geq\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$a+b=ab\Rightarrow b=ab-a=a(b-1)\Rightarrow\frac{1}{a}=\frac{b-1}{b}$$

$$\frac{1}{a^2}+\frac{1}{b^2}=\frac{(b-1)^2}{b^2}+\frac{1}{b^2}\overset{Bergstrom}{\geq}\frac{(b-1+1)^2}{b^2+b^2}=\frac{b^2}{2b^2}=\frac{1}{2}$$

Equality holds for $a=b=2$.

1321. If $a,b>0$ and $ab(a^2+b^2)\geq 2$, then prove that:

$$a^3+b^3\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Ravi Prakash-India

Let $a=r\cos\theta, b=r\sin\theta, r>0, 0<\theta<\frac{\pi}{2}$

$$ab(a^2+b^2)\geq 2\Rightarrow r^4\sin\theta\cos\theta\geq 2\Rightarrow r^4\geq\frac{4}{\sin(2\theta)}\geq 4\Rightarrow r\geq\sqrt{2}$$

Now, $a^3+b^3=r^3(\cos^3\theta+\sin^3\theta)$. Let $f(\theta)=\cos^3\theta+\sin^3\theta, 0<\theta<\frac{\pi}{2}$

$$f'(\theta)=3\cos\theta\sin\theta(\sin\theta-\cos\theta)$$

$$f'(\theta)<0 \text{ if } 0<\theta<\frac{\pi}{4}$$

$$=0 \text{ if } \theta=\frac{\pi}{4}$$

$$>0 \text{ if } \frac{\pi}{4}<\theta<\frac{\pi}{2}$$

$$\therefore \min(f(\theta))=f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Thus, $a^3+b^3\geq 2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)=2$

Equality when $a=b=1$.

1322. If $x, y, z > 0$ and $(x-1)^2 + \left(y - \frac{3}{2}\right)^2 + z^2 \leq \frac{9}{4}$

$$\frac{x^3+x^2+36}{2(x+1)} + \frac{y^3+y^2+36}{4(y+1)} + \frac{2z^3+z^2+9}{2z+1} \geq 16$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$4 \cdot \frac{x^3+x^2+36}{2(x+1)} + 4 \cdot \frac{y^3+y^2+36}{4(y+1)} + 4 \cdot \frac{2z^3+z^2+9}{2z+1} \geq 64$$

$$\left(2x^2 + \frac{144}{2(x+1)}\right) + \left(y^2 + \frac{36}{y+1}\right) + \left(4z^2 + \frac{36}{2z+1}\right) \geq 64$$

$$LHS: \frac{x^2}{1} + \frac{x^2}{1} + \frac{y^2}{1} + \frac{(2z)^2}{1} + \frac{12^2}{2(x+1)} + \frac{6^2}{y+1} + \frac{6^2}{2z+1} \overset{Begstrom}{\geq} \frac{(2x+y+2z+24)^2}{2x+2y+2z+8}$$

Let's prove:

Let $2x + y + 2z = a > 0$ *.Then* $\frac{(a+24)^2}{a+8} \geq 64$

$$(a+24)^2 \geq 64(a+8) \Rightarrow a^2 + 48a + 576 - 64a - 512 \geq 0 \Rightarrow$$

$$\Rightarrow a^2 - 16a + 64 \geq 0 \Rightarrow (a-8)^2 \geq 0 \ (proved)$$

Equality holds if $x = 2, y = 2, z = 1 \Rightarrow (2; 2; 1)$

1323. If $a, b, c \in [0, 1]$, then prove that :

$$\frac{a+1}{b+2} + \frac{b+1}{c+2} + \frac{c+1}{a+2} \leq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Case 1 $a = b = c = 0$ and then : $LHS = 3.\frac{1}{2} = \frac{3}{2} < 2 \therefore \sum_{cyc} \frac{a+1}{b+2} < 2$

Case 2 Exactly 2 variables equal zero and WLOG we may assume : $b = c = 0$ and then : $LHS = \frac{a+1}{2} + \frac{1}{2} + \frac{1}{a+2} \overset{?}{<} 2 \Leftrightarrow \frac{(a+1)(a+2)+2}{2(a+2)} \overset{?}{<} \frac{3}{2}$

$\Leftrightarrow a^2+3a+4\overset{?}{<}3a+6\Leftrightarrow a\overset{?}{<}\sqrt{2}\rightarrow$ true $\because 0\le a\le 1<\sqrt{2}$

$\therefore \frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}<2$

Case 3 Exactly 1 variables equals zero and WLOG we may assume $a=0$ and then : LHS $=\frac{1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{2}$

$=\frac{2c+4+2(b+1)(b+2)+(b+2)(c+2)(c+1)}{2(b+2)(c+2)}\overset{?}{\le}2\Leftrightarrow$

$bc+4\overset{?}{\ge}bc^2+2b^2+2c^2\rightarrow$ true $\because 0\le b,c\le 1\Rightarrow bc\ge bc^2, 2\ge 2b^2$ and $2\ge 2c^2$

$\therefore \frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\le 2,''='' $ iff $(a=0,b=c=1)$ and permutations

Case 4 $a,b,c>0$ and then : $\left(\frac{1}{a}-1\right),\left(\frac{1}{b}-1\right),\left(\frac{1}{c}-1\right)\ge 0$ and denoting $\frac{1}{a}-1$

$=x,\frac{1}{b}-1=y,\frac{1}{c}-1=z$, we get : $a=\frac{1}{x+1},b=\frac{1}{y+1},c=\frac{1}{z+1}$

Now, following simplification, $\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\le 2$ is equivalent to :

$\sum_{cyc}ab+4+2abc\ge\sum_{cyc}ab^2+2\sum_{cyc}a^2$

$\Leftrightarrow\sum_{cyc}\left(\frac{1}{x+1}\cdot\frac{1}{y+1}\right)+4+2\left(\frac{1}{x+1}\cdot\frac{1}{y+1}\cdot\frac{1}{z+1}\right)\ge$

$\sum_{cyc}\left(\frac{1}{x+1}\cdot\left(\frac{1}{y+1}\right)^2\right)+2\sum_{cyc}\left(\frac{1}{x+1}\right)^2$

$\Leftrightarrow\frac{\sum_{cyc}x+3+4\prod_{cyc}(x+1)+2}{\prod_{cyc}(x+1)}\ge$

$\frac{\sum_{cyc}\left((x+1)^2(y+1)\right)+2\sum_{cyc}((x+1)^2(y+1)^2)}{\prod_{cyc}(x+1)^2}$

$\Leftrightarrow 4x^2y^2z^2+8xyz\left(\sum_{cyc}xy\right)+2\sum_{cyc}x^2y^2+17xyz\left(\sum_{cyc}x\right)+4\sum_{cyc}x^2y+5\sum_{cyc}xy^2$

$+40xyz+13\sum_{cyc}xy+3\sum_{cyc}x\ge 0\rightarrow$ true $\because x,y,z\ge 0,''='' $ iff $x=y=z=0\Rightarrow$ iff

$a=b=c=1\therefore$ combining *all cases*, $\frac{a+1}{b+2}+\frac{b+1}{c+2}+\frac{c+1}{a+2}\le 2\ \forall\ a,b,c\in[0,1]$,

$''='' $ iff $(a=0,b=c=1)$ or $(b=0,a=c=1)$ or $(c=0,a=b=1)$

or $(a=b=c=1)$ (QED)

1324. If $a,b,c>0,\ a+b+c=a^2b^2+b^2c^2+c^2a^2$ then:

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\forall\, x,y,z>0\,,\ \sum x^2y^2\geq xyz(x+y+z)$$

$$a+b+c=a^2b^2+b^2c^2+c^2a^2\geq abc(a+b+c) or\ abc\leq 1$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\overset{Am-Gm}{\geq} 3\left(\frac{1}{abc}\right)^{\frac{2}{3}}\geq 3(since\ abc\leq 1)$$

$$Equality\ holds\ for\ a=b=c=1$$

1325. If $a,b,c>0,\ a^3b^3+b^3c^3+c^3a^3=(abc)^4$ then:

$$\frac{1}{a^6}+\frac{1}{b^6}+\frac{1}{c^6}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^3b^3+b^3c^3+c^3a^3=(abc)^4$$

$$(abc)^4\overset{Am-Gm}{\geq} 3(abc)^2\ or\ (abc)^2\geq 3$$

$$\frac{1}{a^6}+\frac{1}{b^6}+\frac{1}{c^6}=\sum\left(\frac{1}{a^3}\right)^2\geq\frac{1}{3}\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)^2=$$

$$=\frac{1}{3}\left(\frac{\sum a^3b^3}{(abc)^3}\right)^2=\frac{1}{3}\left(\frac{(abc)^4}{(abc)^3}\right)^2=\frac{1}{3}(abc)^2\geq\frac{1}{3}.3=1$$

$$Equality\ for\ a=b=c=\sqrt[6]{3}$$

1326. If $a,b\in\left[\frac{1}{2},1\right]$, then prove that :

$$a^5b+ab^5+\frac{6}{a^2+b^2}-3(a+b)\geq -1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a,b\in\left[\frac{1}{2},1\right]\Rightarrow (a-1),(b-1)\leq 0\Rightarrow (a-1)(b-1)\geq 0$$
$$\Rightarrow ab\geq a+b-1\rightarrow (1)$$

$$\text{Now, } a^5b+ab^5+\frac{6}{a^2+b^2}-3(a+b)+1$$
$$=ab(a^4+b^4)+\frac{6}{(a+b)^2-2ab}-3(a+b)+1$$
$$\overset{\text{Holder}}{\geq}\frac{ab(a+b)^4}{8}+\frac{6}{(a+b)^2-2ab}-3(a+b)+1$$
$$\overset{\text{via (1)}}{\geq}\frac{(a+b-1)(a+b)^4}{8}+\frac{6}{(a+b)^2-2(a+b-1)}-3(a+b)+1$$
$$=\frac{(t-1)t^4}{8}+\frac{6}{t^2-2t+2}-3t+1\ (t=a+b)$$
$$=\frac{(t-1)(t^2-2t+2)t^4+48-24t(t^2-2t+2)+8(t^2-2t+2)}{8(t^2-2t+2)}$$
$$=\frac{t^7-3t^6+4t^5-2t^4-24t^3+56t^2-64t+64}{8(t^2-2t+2)}$$
$$=\frac{(t-2)^2(t^5+t^4+4t^3+10t^2+16)}{8(t^2-2t+2)}\geq 0$$
$$\because t=a+b\geq\frac{1}{2}+\frac{1}{2}>0$$

$$\text{and } t^2-2t+2=(t-1)^2+1>0\therefore a^5b+ab^5+\frac{6}{a^2+b^2}-3(a+b)\geq -1$$

$$\forall\ a,b\in\left[\frac{1}{2},1\right],''='' \text{ iff } a=b=1\ (\text{QED})$$

1327. If $a,b,c>0,\ a^2+b^2+c^2=1$ then:

$$\frac{a^5-2a^3+a}{b^2+c^2}+\frac{b^5-2b^3+b}{c^2+a^2}+\frac{c^5-2c^3+c}{a^2+b^2}\leq\frac{2\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a^2+b^2+c^2)=1$$

By Cebyshev: $a^3+b^3+c^3\geq\frac{1}{3}(a^2+b^2+c^2)(a+b+c)=\frac{1}{3}(a+b+c)\ (1)$

By CBS: $(a+b+c)=\sqrt{(a+b+c)^2}\leq\sqrt{3(a^2+b^2+c^2)}=\sqrt{3}\ \ (2)$

$$\frac{a^5-2a^3+a}{b^2+c^2}+\frac{b^5-2b^3+b}{c^2+a^2}+\frac{c^5-2c^3+c}{a^2+b^2}=$$

$$=\sum\frac{a^5-2a^3+a}{b^2+c^2}=\sum\frac{a(a^4-2a^2+1)}{1-a^2}=$$

$$=\sum\frac{a(a^2-1)^2}{1-a^2}=\sum a(1-a^2)=(a+b+c)-(a^3+b^3+c^3)\overset{(1)}{\leq}$$

$$\leq\sum a-\frac{1}{3}(a+b+c)=\frac{2}{3}\sum a\overset{(2)}{\leq\leq}\frac{2\sqrt{3}}{3}$$

$$Equality\ a=b=c=\frac{1}{\sqrt{3}}.$$

1328. If $a,b,c>0$, then prove that :

$$\sum_{cyc}\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\cdot\frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\cdot\frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b}(\sqrt{b}+\sqrt{c})+\sqrt{a}(\sqrt{c}+\sqrt{a})}+$$

$$\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\cdot\frac{\sqrt{b(a+b)}+\sqrt{c(c+a)}}{\sqrt{b}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{c}+\sqrt{a})}+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\cdot\frac{\sqrt{a(a+b)}+\sqrt{c(b+c)}}{\sqrt{a}(\sqrt{a}+\sqrt{b})+\sqrt{c}(\sqrt{b}+\sqrt{c})}\geq$$

$$\overset{CBS}{\geq}\frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}\cdot\frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b}.\sqrt{2(b+c)}+\sqrt{a}.\sqrt{2(c+a)}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{a}}\cdot\frac{\sqrt{b(a+b)}+\sqrt{c(c+a)}}{\sqrt{b}.\sqrt{2(a+b)}+\sqrt{c}.\sqrt{2(c+a)}}$$

$$+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{b}}\cdot\frac{\sqrt{a(a+b)}+\sqrt{c(b+c)}}{\sqrt{a}.\sqrt{2(a+b)}+\sqrt{c}.\sqrt{2(b+c)}}=$$

$$=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2c}}\cdot\frac{\sqrt{b(b+c)}+\sqrt{a(c+a)}}{\sqrt{b(b+c)}+\sqrt{a(c+a)}}+\frac{\sqrt{b}+\sqrt{c}}{\sqrt{2a}}\cdot\frac{\sqrt{b(a+b)}+\sqrt{c(c+a)}}{\sqrt{b(a+b)}+\sqrt{c(c+a)}}$$
$$+\frac{\sqrt{c}+\sqrt{a}}{\sqrt{2b}}\cdot\frac{\sqrt{a(a+b)}+\sqrt{c(b+c)}}{\sqrt{a(a+b)}+\sqrt{c(b+c)}}\overset{A-G}{\geq}$$

$$\geq 3.\sqrt[3]{\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2c}}\cdot\frac{\sqrt{b}+\sqrt{c}}{\sqrt{2a}}\cdot\frac{\sqrt{c}+\sqrt{a}}{\sqrt{2b}}}\overset{Cesaro}{\geq} 3.\sqrt[3]{\frac{8(\sqrt{abc})}{2\sqrt{2}.(\sqrt{abc})}}=3\sqrt{2}$$

$\forall a,b,c>0,''=''$ iff $a=b=c=$ (QED)

1329. If $a,b,c>0$, then prove that :

$$\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\cdot\frac{\sqrt{a(b^2+c^2)}+\sqrt{b(c^2+a^2)}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\cdot\frac{\sqrt{a(b^2+c^2)}+\sqrt{b(c^2+a^2)}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)=$$
$$=\sum_{cyc}\left(\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right).\left(\sqrt{\frac{b^2+c^2}{b}}+\sqrt{\frac{c^2+a^2}{a}}\right)\right)=$$
$$=\sum_{cyc}\left(\left(\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right).\left(\sqrt{\frac{b}{c}+\frac{c}{b}}+\sqrt{\frac{c}{a}+\frac{a}{c}}\right)\right)\geq$$
$$\overset{A-G}{\geq} 2\sqrt{2}.\sum_{cyc}\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}=2\sqrt{2}.\sum_{cyc}\frac{x}{y+z}$$
$$\left(x=\sqrt{c}(\sqrt{a}+\sqrt{b}),y=\sqrt{a}(\sqrt{b}+\sqrt{c}),z=\sqrt{b}(\sqrt{c}+\sqrt{a})\right)\overset{Nesbitt}{\geq} 2\sqrt{2}.\frac{3}{2}=3\sqrt{2}$$

$$\therefore\sum_{cyc}\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}}\cdot\frac{\sqrt{a(b^2+c^2)}+\sqrt{b(c^2+a^2)}}{\sqrt{a}(\sqrt{b}+\sqrt{c})+\sqrt{b}(\sqrt{c}+\sqrt{a})}\right)\geq 3\sqrt{2}\ \forall a,b,c>0,$$

$''=''$ iff $a=b=c$ (QED)

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1330. If $a,b,c>0$, then prove that :

$$\sum_{cyc}\sqrt{\frac{b+c}{a+b}}\cdot\frac{\sqrt{b+c}+\sqrt{a+c}}{\sqrt{(a+b)(b+c)}+(a+c)}\geq\frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution 1 by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall X,Y,Z>0,\ \sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{X^2Y^2}{XY(Y+Z)(Z+X)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}XY\right)^2}{\sum_{cyc}\left(XY\left(\sum_{cyc}XY+Z^2\right)\right)}=\frac{\left(\sum_{cyc}XY\right)^2}{\left(\sum_{cyc}XY\right)^2+XYZ\sum_{cyc}X}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}XY\right)^2\overset{?}{\geq}3XYZ\sum_{cyc}X\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\sqrt{\frac{b+c}{a+b}}\cdot\frac{\sqrt{b+c}+\sqrt{a+c}}{\sqrt{(a+b)(b+c)}+(a+c)}+$$

$$\sqrt{\frac{a+c}{b+c}}\cdot\frac{\sqrt{a+c}+\sqrt{a+b}}{\sqrt{(b+c)(a+c)}+(a+b)}+\sqrt{\frac{a+b}{a+c}}\cdot\frac{\sqrt{a+b}+\sqrt{b+c}}{\sqrt{(a+c)(a+b)}+(b+c)}$$

$$=\frac{x}{z}\cdot\frac{x+y}{zx+y^2}+\frac{y}{x}\cdot\frac{y+z}{xy+z^2}+\frac{z}{y}\cdot\frac{z+x}{yz+x^2}\ \left(x=\sqrt{b+c},y=\sqrt{c+a},z=\sqrt{a+b}\right)$$

$$=\frac{x^2y(x+y)}{xyz(zx+y^2)}+\frac{y^2z(y+z)}{yzx(xy+z^2)}+\frac{z^2x(z+x)}{zxy(yz+x^2)}$$

$$=\frac{x^2y}{z^2x+y^2z}\cdot\left(\frac{x+y}{xy}\right)+\frac{y^2z}{x^2y+z^2x}\cdot\left(\frac{y+z}{yz}\right)+\frac{z^2x}{y^2z+x^2y}\cdot\left(\frac{z+x}{zx}\right)$$

$$=\frac{y^2z}{z^2x+x^2y}\cdot\left(\frac{1}{y}+\frac{1}{z}\right)+\frac{z^2x}{x^2y+y^2z}\cdot\left(\frac{1}{z}+\frac{1}{x}\right)+\frac{x^2y}{y^2z+z^2x}\cdot\left(\frac{1}{x}+\frac{1}{y}\right)$$

$$=\frac{X}{Y+Z}(B+C)+\frac{Y}{Z+X}(C+A)+\frac{Z}{X+Y}(A+B)$$

$$\left(X=y^2z,Y=z^2x,Z=x^2y,A=\frac{1}{x},B=\frac{1}{y},C=\frac{1}{z}\right)$$

$$=\frac{X}{Y+Z}\cdot\sqrt{B+C}^2+\frac{Y}{Z+X}\cdot\sqrt{C+A}^2+\frac{Z}{X+Y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{XY}{(Y+Z)(Z+X)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\frac{1}{xy}}\overset{\text{Bergstrom}}{\geq}\sqrt{\frac{27}{\sum_{cyc}xy}}$$

$$\geq\sqrt{\frac{27}{\sum_{cyc}x^2}}=\sqrt{\frac{27}{\sum_{cyc}(b+c)}}=\frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}$$

$$\therefore\sqrt{\frac{b+c}{a+b}}\cdot\frac{\sqrt{b+c}+\sqrt{a+c}}{\sqrt{(a+b)(b+c)}+(a+c)}+\sqrt{\frac{a+c}{b+c}}\cdot\frac{\sqrt{a+c}+\sqrt{a+b}}{\sqrt{(b+c)(a+c)}+(a+b)}$$

$$+\sqrt{\frac{a+b}{a+c}}\cdot\frac{\sqrt{a+b}+\sqrt{b+c}}{\sqrt{(a+c)(a+b)}+(b+c)}\geq\frac{3\sqrt{3}}{\sqrt{2(a+b+c)}}$$

$$\forall a,b,c>0,''=''\ \text{iff}\ a=b=c\ (\text{QED})$$

Solution 2 by Tapas Das-India

$$Let\ a+b=x,b+c=y,c+a=z,then\ the\ given\ problem\ written\ as$$

$$\sqrt{\frac{y}{x}}\cdot\frac{\sqrt{y}+\sqrt{z}}{\sqrt{xy}+z}+\sqrt{\frac{z}{y}}\cdot\frac{\sqrt{x}+\sqrt{z}}{\sqrt{yz}+x}+\sqrt{\frac{x}{z}}\cdot\frac{\sqrt{y}+\sqrt{x}}{\sqrt{xz}+y}\geq\frac{3\sqrt{3}}{\sqrt{x+y+z}}$$

$$Now\ \sqrt{\frac{y}{x}}\cdot\frac{\sqrt{y}+\sqrt{z}}{\sqrt{xy}+z}=\frac{y+\sqrt{yz}}{\sqrt{x}(z+\sqrt{xy})},\sqrt{\frac{z}{y}}\cdot\frac{\sqrt{x}+\sqrt{z}}{\sqrt{yz}+x}=\frac{z+\sqrt{xz}}{\sqrt{y}(x+\sqrt{yz})},\sqrt{\frac{x}{z}}\cdot\frac{\sqrt{y}+\sqrt{x}}{\sqrt{xz}+y}$$

$$=\frac{x+\sqrt{yx}}{\sqrt{z}(\sqrt{xz}+y)},Now\sum\sqrt{\frac{y}{x}}\cdot\frac{\sqrt{y}+\sqrt{z}}{\sqrt{xy}+z}$$

$$=\sum\frac{y+\sqrt{yz}}{\sqrt{x}(z+\sqrt{xy})}\overset{AM-GM}{\geq}3\sqrt[3]{\prod\frac{y+\sqrt{yz}}{\sqrt{x}(z+\sqrt{xy})}}$$

$$=3\sqrt[3]{\frac{1}{\sqrt{xyz}}}\overset{AM-GM}{\geq}\frac{3}{\frac{\sqrt{x}+\sqrt{y}+\sqrt{z}}{3}}\overset{CBS}{\geq}\frac{9}{\sqrt{3(x+y+z)}}$$

$$=\frac{3\sqrt{3}}{\sqrt{x+y+z}}\ Equality\ holds\ for\ x=y=z\ or\ a=b=c$$

1331. If $a=\frac{\varphi^2}{2(1+\varphi)},b=\frac{\varphi}{2(1+\varphi)},c=\frac{1}{2(1+\varphi)},\varphi-$ golden ratio, then

$$\sum_{cyc}\frac{a^2b^2}{c^3(a^2-ab+b^2)}>6\varphi$$

Proposed by Khaled Abd Imouti-Syria

Solution by Izumi Ainsworth-Japan

$$a=\frac{\varphi^2}{2(\varphi+1)}=\frac{1}{2}, b=\frac{\varphi}{2(\varphi+1)}=\frac{1}{2\varphi}, c=\frac{1}{2(1+\varphi)}=\frac{1}{2\varphi^2}$$

$$*\ \frac{a^2b^2}{c^3(a^2-ab+b^2)}=\frac{\left(\frac{1}{2\varphi}\right)^2\left(\frac{1}{2\varphi^2}\right)^2}{\left(\frac{1}{2\varphi}\right)^3\left(\frac{1}{4}-\frac{1}{4\varphi}+\frac{1}{4\varphi^2}\right)}=\frac{2\varphi^4}{\frac{2}{\varphi^2}}=\varphi^6 \quad (\alpha)$$

$$*\ \frac{b^2c^2}{a^3(b^2-bc+c^2)}=\frac{\left(\frac{1}{2\varphi}\right)^2\left(\frac{1}{2\varphi^2}\right)^2}{\left(\frac{1}{2}\right)^3\left(\frac{1}{4\varphi^2}-\frac{1}{4\varphi^3}+\frac{1}{4\varphi^4}\right)}=\frac{2}{\varphi^6\left(\frac{2}{\varphi^4}\right)}=\frac{1}{\varphi^2} \quad (\beta)$$

$$*\ \frac{c^2a^2}{b^3(c^2-ca+a^2)}=\frac{\left(\frac{1}{2\varphi^2}\right)^2\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2\varphi}\right)^3\left(\frac{1}{4\varphi^4}-\frac{1}{4\varphi^4}+\frac{1}{4}\right)}=\frac{2}{\varphi\left(\frac{2}{\varphi^2}\right)}=\frac{1}{\varphi} \quad (\gamma)$$

$(\alpha)+(\beta)+(\gamma)$:

$$\therefore \sum_{cyc}\frac{a^2b^2}{c^3(a^2-ab+b^2)}=\varphi^6+\frac{1}{\varphi^2}+\frac{1}{\varphi}$$

$$=(8\varphi+5)+(2-\varphi)+(\varphi-1)=8\varphi+6>6\varphi$$

$\varphi\to\frac{\sqrt{5}+1}{2}$ **(Golden ratio)**

1332. If $x_i>0(i=1,2,\ldots,n)$ and $k\in\mathbb{N}^*$, then prove that:

$$2\sum_{i=1}^{n}x_i^{2k+1}\geq\sum_{cyclic}x_1x_2\left(x_1^{2k-1}+x_2^{2k-1}\right)\geq\sum_{cyc}x_1^2x_2^2\left(x_1^{2k-3}+x_2^{2k-3}\right)\geq\cdots\geq$$

$$\geq\sum_{cyclic}x_1^kx_2^k(x_1+x_2)$$

Proposed by Neculai Stanciu – Romania

Solution by Eric-Dimitrie Cismaru-Romania

For any $a,b\in\mathbb{R}_+$ and $p\in\mathbb{N}^*$, we have the following inequality:

$$a^{p+1}+b^{p+1}\geq ab(a^{p-1}+b^{p-1})\Leftrightarrow a^p(a-b)-b^p(a-b)=$$

$$=(a-b)^2(a^{p-1}+\cdots+b^{p-1})\geq 0$$

Furthermore, we have $a^{2p+1}+b^{2p+1}\geq ab(a^{2p-1}+b^{2p-1})\geq$

$$\geq a^2b^2(a^{2p-3}+b^{2p-3})\geq\cdots\geq a^pb^p(a+b)$$

Using this result k times, we obtain the desired inequality

$$2\sum_{i=1}^{n} x_i^{2k+1} = \sum_{cyc}\left(x_1^{2k+1} + x_2^{2k+1}\right) \geq \sum_{cyc} x_1x_2\left(x_1^{2k-1} + x_2^{2k-1}\right) \geq$$

$$\geq \sum_{cyc} x_1^2x_2^2\left(x_1^{2k-3} + x_2^{2k-3}\right) \geq \cdots \geq \sum_{cyc} x_1^kx_2^k(x_1 + x_2).$$

Equality holds for $x_1 = x_2 = \cdots = x_n$.

1333. If $a, b, c > 0, abc = 1$, $\lambda \leq \frac{81}{2}$ then:

$$(a+b+c)(ab+bc+ca) + \frac{\lambda}{ab+bc+ca} \geq 9 + \frac{\lambda}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ (ab+bc+ca) = x\ then\ \sum a = \sqrt{\left(\sum a\right)^2} \geq \sqrt{3\sum ab} = \sqrt{3x}$$

$$Now\ we\ need\ to\ show:$$

$$(a+b+c)(ab+bc+ca) + \frac{\lambda}{ab+bc+ca} \geq 9 + \frac{\lambda}{3} or,$$

$$x\sqrt{3x} + \frac{\lambda}{x} \geq 9 + \frac{\lambda}{3}\ or,$$

$$\sqrt{3}u^3 + \frac{\lambda}{u^2} \overset{x=u^2>0}{\geq} 9 + \frac{\lambda}{3} or, 3\sqrt{3}u^5 - u^2(27+\lambda) + 3\lambda \geq 0\ or,$$

$$\left(u-\sqrt{3}\right)\left(3\sqrt{3}u^2\left(u^2+\sqrt{3}u+3\right) - \left(u+\sqrt{3}\right)\lambda\right) \geq 0\ or,$$

$$\left(u-\sqrt{3}\right)\left(3\sqrt{3}u^2\left(u^2+\sqrt{3}u+3\right) - \left(u+\sqrt{3}\right)\frac{81}{2}\right) \overset{\lambda\leq\frac{81}{2}}{\geq} 0\ or,$$

$$\left(u-\sqrt{3}\right)^2\left(6\sqrt{3}u^3 + 36u^2 + 54\sqrt{3}u + 81\right) \geq 0(True)$$

$$Equality\ for\ u = \sqrt{3}\ or\ u^2 = x = ab+bc+ca = 3\ or\ a = b = c = 1$$

1334. If $a,b,c>0, a+b+c=1$, $\lambda \geq 0$ then:

$$\sum_{cyc} \frac{a}{b(b+\lambda c)} \geq \frac{9}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{a}{b(b+\lambda c)} = \sum \frac{\frac{a}{b}}{b+\lambda c} =$$

$$= \sum \frac{\left(\sqrt{\frac{a}{b}}\right)^2}{b+\lambda c} \overset{Bergstrom}{\geq} \frac{\left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{a}}\right)^2}{(\lambda+1)(a+b+c)} \overset{AM-GM}{\geq}$$

$$\geq \frac{(3)^2}{\lambda+1} = \frac{9}{\lambda+1} (since\ \ a+b+c=1)$$

$$Equality\ for\ a=b=c=\frac{1}{3}$$

1335. If $a,b,c>0, \lambda \geq 0$ then:

$$\sum_{cyc} \frac{bc}{a\sqrt{b+\lambda c}} \geq \frac{ab+bc+ca}{\sqrt{(\lambda+1)abc}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \frac{bc}{a\sqrt{b+\lambda c}} = \sum \frac{(bc)^{\frac{3}{2}}}{a\sqrt{bc}\sqrt{b+\lambda c}} =$$

$$= \frac{1}{\sqrt{abc}} \sum \frac{(bc)^{\frac{3}{2}}}{\sqrt{ab+\lambda ac}} \overset{Radon}{\geq} \frac{1}{\sqrt{abc}} \frac{(\sum bc)^{\frac{3}{2}}}{\sqrt{\sum ab+\lambda \sum ac}} = \frac{ab+bc+ca}{\sqrt{(\lambda+1)abc}}$$

$$Equality\ for\ a=b=c=1$$

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1336. If $a,b,c>0$ and $\lambda\geq 0$, then :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}=\sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{a^2+\lambda b^2}{a^3+\lambda b^3}\right)$$

$$=\sum_{cyc}\frac{(b+a)(a^3+\lambda b^3)-2ab(a^2+\lambda b^2)}{2ab(a^3+\lambda b^3)}=\sum_{cyc}\frac{\lambda b^4-\lambda ab^3+a^4-a^3b}{2ab(a^3+\lambda b^3)}$$

$$=\sum_{cyc}\frac{\lambda b^3(b-a)-a^3(b-a)}{2ab(a^3+\lambda b^3)}=\sum_{cyc}\frac{(b-a)(\lambda b^3-a^3)}{2ab(a^3+\lambda b^3)}$$

$$=\sum_{cyc}\frac{(b-a)(\lambda b^3+a^3-2a^3)}{2ab(a^3+\lambda b^3)}=\sum_{cyc}\frac{b-a}{2ab}-\frac{a^3(b-a)}{ab(a^3+\lambda b^3)}$$

$$=\frac{1}{2abc}\cdot\sum_{cyc}c(b-a)-\sum_{cyc}\frac{a^3c(b^3+\lambda c^3)(c^3+\lambda a^3)(b-a)}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}=0-$$

$$\frac{1}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}\cdot\sum_{cyc}\left(a^3c\begin{pmatrix}b^3c^3+\lambda(a^3b^3+c^6)\\+\lambda^2c^3a^3\end{pmatrix}(b-a)\right)$$

$$=-\frac{a^3b^3c^3\sum_{cyc}c(b-a)+\lambda.\sum_{cyc}\left(a^3c(a^3b^3+c^6)(b-a)\right)+\lambda^2.\sum_{cyc}\left(a^6c^4(b-a)\right)}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}.$$

$$=\frac{\lambda\left(\sum_{cyc}a^7b^4-abc\left(\sum_{cyc}a^5b^3\right)\right)+\lambda^2\left(\sum_{cyc}a^4b^7-abc\left(\sum_{cyc}a^3b^5\right)\right)}{abc(a^3+\lambda b^3)(b^3+\lambda c^3)(c^3+\lambda a^3)}$$

$\therefore$ since $\lambda\geq 0$ $\therefore$ in order to prove : $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}$,

it sufficess to prove :

$$\boxed{\sum_{cyc}a^7b^4\overset{(*)}{\geq}abc\left(\sum_{cyc}a^5b^3\right)}\ \text{and}\ \boxed{\sum_{cyc}a^4b^7\overset{(\bullet)}{\geq}abc\left(\sum_{cyc}a^3b^5\right)}$$

$$\text{Now,}\sum_{cyc}a^7b^4=abc\sum_{cyc}\frac{a^6b^3}{c}=abc\sum_{cyc}\frac{a^{10}b^6}{a^4b^3c}\overset{\text{Bergstrom}}{\geq}$$

$$abc.\frac{\left(\sum_{cyc}a^5b^3\right)^2}{abc\left(\sum_{cyc}a^3b^2\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^5b^3\right)\Leftrightarrow\sum_{cyc}a^5b^3\underset{(**)}{\overset{?}{\geq}}abc\left(\sum_{cyc}a^3b^2\right)\ \text{and,}$$

$$\sum_{cyc}a^5b^3=abc\sum_{cyc}\frac{a^4b^2}{c}=abc\sum_{cyc}\frac{a^6b^4}{a^2b^2c}\overset{\text{Bergstrom}}{\geq}$$

$$abc\cdot\frac{\left(\sum_{cyc}a^3b^2\right)^2}{abc\left(\sum_{cyc}ab\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^3b^2\right)\Leftrightarrow\sum_{cyc}a^3b^2\overset{?}{\underset{(***)}{\geq}}abc\left(\sum_{cyc}ab\right)\text{ and,}$$

$$\sum_{cyc}a^3b^2=abc\sum_{cyc}\frac{a^2b}{c}=abc\sum_{cyc}\frac{a^2b^2}{bc}\overset{\text{Bergstrom}}{\geq}abc\cdot\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab}=abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow(***)\Rightarrow(**)\Rightarrow\boxed{(*)\text{ is true}}$$

$$\text{Again, }\sum_{cyc}a^4b^7=abc\sum_{cyc}\frac{a^3b^6}{c}=abc\sum_{cyc}\frac{a^6b^{10}}{a^3b^4c}\overset{\text{Bergstrom}}{\geq}$$

$$abc\cdot\frac{\left(\sum_{cyc}a^3b^5\right)^2}{abc\left(\sum_{cyc}a^2b^3\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^3b^5\right)\Leftrightarrow\sum_{cyc}a^3b^5\overset{?}{\underset{(**)}{\geq}}abc\left(\sum_{cyc}a^2b^3\right)\text{ and,}$$

$$\sum_{cyc}a^3b^5=abc\sum_{cyc}\frac{a^2b^4}{c}=abc\sum_{cyc}\frac{a^4b^6}{a^2b^2c}\overset{\text{Bergstrom}}{\geq}$$

$$abc\cdot\frac{\left(\sum_{cyc}a^2b^3\right)^2}{abc\left(\sum_{cyc}ab\right)}\overset{?}{\geq}abc\left(\sum_{cyc}a^2b^3\right)\Leftrightarrow\sum_{cyc}a^2b^3\overset{?}{\underset{(***)}{\geq}}abc\left(\sum_{cyc}ab\right)\text{ and,}$$

$$\sum_{cyc}a^2b^3=abc\sum_{cyc}\frac{ab^2}{c}=abc\sum_{cyc}\frac{a^2b^2}{ca}\overset{\text{Bergstrom}}{\geq}abc\cdot\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab}=abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow(***)\Rightarrow(**)\Rightarrow\boxed{(\bullet)\text{ is true}}\therefore(*)\text{ and }(\bullet)\text{ are both true}$$

$$\therefore\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^2+\lambda b^2}{a^3+\lambda b^3}\ \forall\ a,b,c>0\text{ and }\lambda\geq0,''='' \text{ iff } a=b=c\ (\text{QED})$$

1337. If $x,y,z>0, x+y+z=1, \lambda\geq 1$ then:

$$\sum_{cyc}\frac{x^2+\lambda xy}{x+y}\leq\frac{\lambda+1}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{x^2+\lambda xy}{x+y}=\sum\left[x+\frac{(\lambda-1)xy}{x+y}\right]=$$

$$=\sum x+(\lambda-1)\sum\frac{xy}{x+y}\overset{Am-Gm}{\leq}1+(\lambda-1)\sum\frac{xy}{2\sqrt{xy}}=$$

$$=1+\frac{(\lambda-1)}{2}\sum\sqrt{xy}\overset{CBS}{\leq}1+\frac{(\lambda-1)}{2}\sqrt{3(xy+yz+zx)}\leq$$

$$\leq 1+\frac{(\lambda-1)}{2}\sqrt{(x+y+z)^2}=1+\frac{(\lambda-1)}{2}=\frac{\lambda+1}{2}$$

$$Equality\ for\ x=y=z=\frac{1}{3}$$

1338. If $a,b,c>0$ and $\lambda\geq 0$, then :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4}=\sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{a^3+\lambda b^3}{a^4+\lambda b^4}\right)$$

$$=\sum_{cyc}\frac{(b+a)(a^4+\lambda b^4)-2ab(a^3+\lambda b^3)}{2ab(a^4+\lambda b^4)}=\sum_{cyc}\frac{\lambda b^5-\lambda ab^4+a^5-a^4b}{2ab(a^4+\lambda b^4)}$$

$$=\sum_{cyc}\frac{\lambda b^4(b-a)-a^4(b-a)}{2ab(a^4+\lambda b^4)}=\sum_{cyc}\frac{(b-a)(\lambda b^4-a^4)}{2ab(a^4+\lambda b^4)}$$

$$=\sum_{cyc}\frac{(b-a)(\lambda b^4+a^4-2a^4)}{2ab(a^4+\lambda b^4)}=\sum_{cyc}\frac{b-a}{2ab}-\sum_{cyc}\frac{a^4(b-a)}{ab(a^4+\lambda b^4)}$$

$$=\frac{1}{2abc}.\sum_{cyc}c(b-a)-\sum_{cyc}\frac{a^4c(b^4+\lambda c^4)(c^4+\lambda a^4)(b-a)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$$=0-\frac{\sum_{cyc}\left(a^4c(b^4c^4+\lambda(a^4b^4+c^8)+\lambda^2c^4a^4)(b-a)\right)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$$=-\frac{a^4b^4c^4\sum_{cyc}c(b-a)+\lambda.\sum_{cyc}\left(a^4c(a^4b^4+c^8)(b-a)\right)+\lambda^2.\sum_{cyc}\left(a^8c^5(b-a)\right)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$$=\frac{\lambda\left(\sum_{cyc}a^9b^5-abc\left(\sum_{cyc}a^7b^4\right)\right)+\lambda^2\left(\sum_{cyc}a^5b^9-abc\left(\sum_{cyc}a^4b^7\right)\right)}{abc(a^4+\lambda b^4)(b^4+\lambda c^4)(c^4+\lambda a^4)}$$

$\therefore$ since $\lambda\geq 0$ $\therefore$ in order to prove : $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4}$, it suffices

to prove : $\boxed{\sum_{cyc}a^9b^5\overset{(*)}{\geq}abc\left(\sum_{cyc}a^7b^4\right)}$ and $\boxed{\sum_{cyc}a^5b^9\overset{(\bullet)}{\geq}abc\left(\sum_{cyc}a^4b^7\right)}$

$$\text{Now, }\sum_{cyc}a^9b^5=abc\sum_{cyc}\frac{a^8b^4}{c}=abc\sum_{cyc}\frac{a^{14}b^8}{a^6b^4c}\overset{\text{Bergstrom}}{\geq}abc.\frac{\left(\sum_{cyc}a^7b^4\right)^2}{abc\left(\sum_{cyc}a^5b^3\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^7b^4\right) \Leftrightarrow \sum_{cyc} a^7b^4 \underset{(**)}{\overset{?}{\geq}} abc\left(\sum_{cyc} a^5b^3\right) \text{ and,}$$

$$\sum_{cyc} a^7b^4 = abc\sum_{cyc}\frac{a^6b^3}{c} = abc\sum_{cyc}\frac{a^{10}b^6}{a^4b^3c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^5b^3\right)^2}{abc\left(\sum_{cyc} a^3b^2\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^5b^3\right) \Leftrightarrow \sum_{cyc} a^5b^3 \underset{(***)}{\overset{?}{\geq}} abc\left(\sum_{cyc} a^3b^2\right) \text{ and,}$$

$$\sum_{cyc} a^5b^3 = abc\sum_{cyc}\frac{a^4b^2}{c} = abc\sum_{cyc}\frac{a^6b^4}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^2\right)^2}{abc\left(\sum_{cyc} ab\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^3b^2\right) \Leftrightarrow \sum_{cyc} a^3b^2 \underset{(****)}{\overset{?}{\geq}} abc\left(\sum_{cyc} ab\right) \text{ and,}$$

$$\sum_{cyc} a^3b^2 = abc\sum_{cyc}\frac{a^2b}{c} = abc\sum_{cyc}\frac{a^2b^2}{bc} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} ab} = abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow \boxed{(*) \text{ is true}}$$

$$\text{Again, } \sum_{cyc} a^5b^9 = abc\sum_{cyc}\frac{a^4b^8}{c} = abc\sum_{cyc}\frac{a^8b^{14}}{a^4b^6c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^4b^7\right)^2}{abc\left(\sum_{cyc} a^3b^5\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^4b^7\right) \Leftrightarrow \sum_{cyc} a^4b^7 \underset{(\bullet\bullet)}{\overset{?}{\geq}} abc\left(\sum_{cyc} a^3b^5\right) \text{ and,}$$

$$\sum_{cyc} a^4b^7 = abc\sum_{cyc}\frac{a^3b^6}{c} = abc\sum_{cyc}\frac{a^6b^{10}}{a^3b^4c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^5\right)^2}{abc\left(\sum_{cyc} a^2b^3\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^3b^5\right) \Leftrightarrow \sum_{cyc} a^3b^5 \underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}} abc\left(\sum_{cyc} a^2b^3\right) \text{ and,}$$

$$\sum_{cyc} a^3b^5 = abc\sum_{cyc}\frac{a^2b^4}{c} = abc\sum_{cyc}\frac{a^4b^6}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^2b^3\right)^2}{abc\left(\sum_{cyc} ab\right)}$$

$$\overset{?}{\geq} abc\left(\sum_{cyc} a^2b^3\right) \Leftrightarrow \sum_{cyc} a^2b^3 \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\geq}} abc\left(\sum_{cyc} ab\right) \text{ and,}$$

$$\sum_{cyc} a^2b^3 = abc\sum_{cyc}\frac{ab^2}{c} = abc\sum_{cyc}\frac{a^2b^2}{ca} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} ab} = abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow \boxed{(\bullet) \text{ is true}} \therefore (*) \text{ and } (\bullet) \text{ are both true}$$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \sum_{cyc}\frac{a^3+\lambda b^3}{a^4+\lambda b^4} \forall\, a,b,c>0 \text{ and } \lambda \geq 0, \text{''} = \text{''} \text{ iff } a=b=c \text{ (QED)}$$

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1339. If $x,y,z>0$ and $x+y+z=3$ with $\lambda\geq 2$, then :

$$\sum_{cyc}\frac{x^2}{x+\lambda y^2}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{x^2}{x+\lambda y^2}=\sum_{cyc}\frac{x^4}{x^3+\lambda x^2y^2}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x^2\right)^2}{\sum_{cyc}x^3+\lambda\sum_{cyc}x^2y^2}\overset{x+y+z=3}{=}$$

$$\frac{\left(\sum_{cyc}x^2\right)^2}{\frac{\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)}{3}+\lambda\sum_{cyc}x^2y^2}=\frac{3\left(\sum_{cyc}x^2\right)^2}{\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)+3\lambda\sum_{cyc}x^2y^2}\overset{?}{\geq}\frac{3}{\lambda+1}$$

$$\Leftrightarrow\lambda\left(\left(\sum_{cyc}x^2\right)^2-3\sum_{cyc}x^2y^2\right)+\left(\sum_{cyc}x^2\right)^2-\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x\right)\overset{(*)}{\boxed{\geq}}\overset{?}{0}$$

Assigning $y+z=a, z+x=b, x+y=c\Rightarrow a+b-c=2z>0, b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c, b+c>a, c+a>b\Rightarrow a,b,c$ form sides of a triangle with semiperimeter, circumradius and inradius $=s,R,r$ (say)

yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x\overset{(\bullet)}{=}s\Rightarrow x=s-a, y=s-b, z=s-c$

$$\therefore xyz\overset{(\bullet\bullet)}{=}r^2s \text{ and, } \sum_{cyc}xy=\sum_{cyc}(s-a)(s-b)=4Rr+r^2\Rightarrow\sum_{cyc}xy\overset{(\bullet\bullet\bullet)}{=}4Rr+r^2$$

$$\text{and also, } \sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\overset{\text{via }(\bullet)\text{ and }(\bullet\bullet\bullet)}{=}s^2-2(4Rr+r^2)\Rightarrow$$

$$\sum_{cyc}x^2\overset{(\bullet\bullet\bullet\bullet)}{=}s^2-8Rr-2r^2 \text{ and also, } \sum_{cyc}x^2y^2=\left(\sum_{cyc}xy\right)^2-2xyz\left(\sum_{cyc}x\right)$$

$$\overset{\text{via }(\bullet),(\bullet\bullet)\text{ and }(\bullet\bullet\bullet)}{=}(4Rr+r^2)^2-2r^2s^2\rightarrow(\bullet\bullet\bullet\bullet\bullet) \text{ and } \sum_{cyc}x^3=$$

$$\left(\sum_{cyc}x\right)^3-3(x+y)(y+z)(z+x)\overset{\text{via }(\bullet)}{=}s^3-12Rrs\rightarrow(\bullet\bullet\bullet\bullet\bullet\bullet)$$

$$\therefore via\ (\bullet\bullet\bullet\bullet),(\bullet\bullet\bullet\bullet\bullet)\ and\ (\bullet\bullet\bullet\bullet\bullet\bullet)\ and\ \because \lambda \geq 2\ and \left(\sum_{cyc} x^2\right)^2 - 3\sum_{cyc} x^2y^2 \geq 0,$$

$$\text{LHS of } (*) \geq 2\left((s^2-8Rr-2r^2)^2 - 3((4Rr+r^2)^2 - 2r^2s^2)\right) + (s^2-8Rr-2r^2)^2$$

$$-s(s^3-12Rrs) \overset{?}{\geq} 0 \Leftrightarrow s^4 - 18Rrs^2 + 3r^2(4R+r)^2 \boxed{\overset{?}{\geq}_{(**)}} 0\ and$$

$$\because (s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0\ \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$\text{LHS of } (**) \geq (s^2-16Rr+5r^2)^2 \Leftrightarrow (7R-5r)s^2 \boxed{\overset{?}{\geq}_{(***)}} r(104R^2-92Rr+11r^2)$$

$$\text{Now, } (7R-5r)s^2 \overset{\text{Gerretsen}}{\geq} (7R-5r)(16Rr-5r^2) \overset{?}{\geq} r(104R^2-92Rr+11r^2)$$

$$\Leftrightarrow 8R^2-23Rr+14r^2 \overset{?}{\geq} 0 \Leftrightarrow (8R-7r)(R-2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{x^2}{x+\lambda y^2} \geq \frac{3}{\lambda+1}\ \forall x,y,z>0 \,\big|\, x+y+z=3$$

$$\text{with } \lambda \geq 2, ''='' \text{ iff } x=y=z=1\ (\text{QED})$$

1340. If $a,b,c>0$ and $\lambda \geq \frac{1}{4}$, then :

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{ab+bc+ca} \geq \lambda+\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{ab+bc+ca}-\lambda-\frac{3}{2}=$$

$$=\sum_{cyc}\frac{a^2}{a^2+ab}-\frac{3}{2}+\frac{\lambda\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}{\sum_{cyc}ab} \geq$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{\sum_{cyc}a^2+\sum_{cyc}ab}-\frac{3}{2}+\frac{\lambda\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)}{\sum_{cyc}ab} \overset{\lambda\geq\frac{1}{4}}{\geq}$$

$$\geq \frac{y-x}{2(x+y)}+\frac{x-y}{4y}\left(x=\sum_{cyc}a^2\ \text{and}\ y=\sum_{cyc}ab\right)=$$

$$=\left(\frac{x-y}{2}\right)\left(\frac{1}{2y}-\frac{1}{x+y}\right)=\frac{(x-y)^2}{4y(x+y)} \geq 0$$

$$\therefore \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{ab+bc+ca}\geq \lambda+\frac{3}{2}\ \forall\, a,b,c>0\,\big|\,\lambda\geq\frac{1}{4},$$
$$''='' \text{ iff } a=b=c \text{ (QED)}$$

1341. If $a,b>0$ then:

$$\frac{a^2}{b}+\frac{b^2}{a}+4\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$We\ need\ to\ show$:

$$\frac{a^2}{b}+\frac{b^2}{a}+4\left(\frac{1}{a+1}+\frac{1}{b+1}\right)\geq 6\ or,$$

$$\frac{(a+b)^2}{a+b}+\frac{4(1+1)^2}{a+b+2}\geq 6(Bergstrom)$$

$$or\ x+\frac{16}{x+2}\overset{a+b=x>0}{\geq} 6\ or$$

$$16\geq(6-x)(x+2)or,$$

$$16\geq 4x+12-x^2\ or,$$

$$x^2-4x+4\geq 0\ or$$

$$(x-2)^2\geq 0\ true.$$

$$Equality\ for\ \ x=a+b=2\ or\ a=b=1$$

1342. If $a,b,c,d\in R,\ \ a+b+c+d=4$ then:

$$(a^4+3)(b^4+3)(c^4+3)(d^4+3)\geq 256$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$(a^4+3)(b^4+3)(c^4+3)(d^4+3)=$$

$$(a^4+1+1+1)(1+b^4+1+1)(1+1+c^4+1)(1+1+1+d^4) \overset{Holder}{\geq}$$

$$\geq (a+b+c+d)^4 = 4^4 = 256$$

$$Equality\ for\ a=b=c=d=1$$

1343. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$\frac{2}{3+ab+bc+ca}+\sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}}\leq\frac{5}{6}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{5}{6}-\frac{2}{3+\sum_{cyc}ab}=\frac{15+5\sum_{cyc}ab-12}{6\left(3+\sum_{cyc}ab\right)}\Rightarrow\frac{5}{6}-\frac{2}{3+\sum_{cyc}ab}$$

$$=\frac{3+5x}{6(3+x)}\to(1)\left(x=\sum_{cyc}ab\right)$$

$$\text{Now},\frac{abc}{(1+a)(1+b)(1+c)}=\frac{abc}{1+abc+\sum_{cyc}ab+\sum_{cyc}a}$$

$$=\frac{abc+1+\sum_{cyc}ab+\sum_{cyc}a-\left(1+\sum_{cyc}ab+\sum_{cyc}a\right)}{1+abc+\sum_{cyc}ab+\sum_{cyc}a}$$

$$=1-\frac{1+\sum_{cyc}ab+\sum_{cyc}a}{1+abc+\sum_{cyc}ab+\sum_{cyc}a}\leq 1-\frac{1+\sum_{cyc}ab+\sum_{cyc}a}{1+\frac{\left(\sum_{cyc}ab\right)^2}{9}+\sum_{cyc}ab+\sum_{cyc}a}$$

$$\left(\because 3abc\left(\sum_{cyc}a\right)\leq\left(\sum_{cyc}ab\right)^2\overset{a+b+c=3}{\Rightarrow}abc\leq\frac{1}{9}\left(\sum_{cyc}ab\right)^2\right)$$

$$=\frac{\frac{\left(\sum_{cyc}ab\right)^2}{9}}{1+\frac{\left(\sum_{cyc}ab\right)^2}{9}+\sum_{cyc}ab+\sum_{cyc}a}\overset{a+b+c=3}{=}\frac{x^2}{9+x^2+9x+27}$$

$$\Rightarrow\sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}}\leq\sqrt[3]{\frac{x^2}{36+x^2+9x}}\overset{?}{\leq}\frac{3+5x}{6(3+x)}$$

$$\Leftrightarrow\frac{(3+5x)^3}{216(3+x)^3}\overset{?}{\geq}\frac{x^2}{36+x^2+9x}\Leftrightarrow(36+x^2+9x)(3+5x)^3\overset{?}{\geq}216x^2(3+x)^3$$

$$\Leftrightarrow -91x^5-594x^4+828x^3+3510x^2+5103x+972\overset{?}{\geq}0$$

$$\Leftrightarrow(3-x)(91x^4+867x^3+1773x^2+1809x+324)\overset{?}{\geq}0\to\text{true}$$

$$\because x=\sum_{cyc} ab>0 \; and \; 3x\le\left(\sum_{cyc} a\right)^2=9\Rightarrow 3-x\ge 0$$

$$\therefore \sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}}\le\frac{3+5x}{6(3+x)}\overset{via\,(1)}{=}\frac{5}{6}-\frac{2}{3+\sum_{cyc} ab}$$

$$\Rightarrow\frac{2}{3+ab+bc+ca}+\sqrt[3]{\frac{abc}{(1+a)(1+b)(1+c)}}\le\frac{5}{6}$$

$$\forall a,b,c>0\,|\,a+b+c=3,''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1344. If $a,b,c>0$ then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge\frac{a^2+2b^2}{a^3+2b^3}+\frac{b^2+2c^2}{b^3+2c^3}+\frac{c^2+2a^2}{c^3+2a^3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$a^3+2b^3=a^3+b^3+b^3\ge\frac{1}{3}(a+b+b)(a^2+b^2+b^2)=$$

$$=\frac{1}{3}(a+b+b)(a^2+2b^2)\;(1)$$

$$\frac{a^2+2b^2}{a^3+2b^3}+\frac{b^2+2c^2}{b^3+2c^3}+\frac{c^2+2a^2}{c^3+2a^3}=\sum\frac{a^2+2b^2}{a^3+2b^3}\overset{(1)}{\le}$$

$$\le 3\sum\frac{1}{a+b+b}\overset{AM-HM}{\le}\frac{1}{3}\sum\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

$$Equality\ for\ a=b=c=1$$

1345. If $a,b,c\ge 0$, $a^2+b^2+c^2=1$ then:

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}\ge\frac{4}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}=\sum\frac{1}{a^3+2}=$$

$$=\frac{1}{2}\sum\frac{2}{a^3+2}=\frac{1}{2}\sum\left(1-\frac{a^3}{a^3+1+1}\right)\overset{AM-GM}{\geq}\frac{3}{2}-\frac{1}{2}\sum\frac{a^3}{3a}=$$

$$=\frac{3}{2}-\frac{1}{6}\sum a^2=\frac{3}{2}-\frac{1}{6}=\frac{4}{3}$$

$$Equality\ for\ (a,b,c)=(1,0,0)or(0,1,0)or(0,0,1)$$

1346. If $x,y,z>0,\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$ then:

$$xyz(x+y+z)(x+1)(y+1)(z+1)\geq 1296$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1\ \ (1)\ or$$

$$x+y+z+2=xyz\ (2)$$

$$[(x+1)+(y+1)+(z+1)]\left[\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right]\overset{C-S}{\geq}9\ or$$

$$(x+y+z+3)\overset{(1)}{\geq}9\ or\ x+y+z\geq 6\ (2)$$

$$Now\ from(2)\ xyz\geq 6+2=8\ \ (3)\ and$$

$$xy+yz+zx\overset{AM-GM}{\geq}3(xyz)^{\frac{2}{3}}\overset{(3)}{\geq}3.(8)^{\frac{2}{3}}=12\ \ (4)$$

$$xyz(x+y+z)(x+1)(y+1)(z+1)=$$

$$=xyz(x+y+z)(1+x+y+z+xy+yz+zx+xyz)\overset{(2)\&(3)\&(4)}{\geq}$$

$$\geq 8.6(1+6+12+8)=1296$$

$$Equality\ for\ x=y=z=2$$

1347. If $x,y,z>0, y\in(0,z)\cup[x,\infty), x\geq z$ then:

$$\frac{xz}{y^2+yz}+\frac{y^2}{x^2+yz}+\frac{x+2z}{x+z}\geq\frac{5}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Khaled Abd Imouti-Damascus-Syria

$$\frac{xz}{y^2+yz}+\frac{y^2}{xz+yz}+\frac{yz}{xz+y^2}\overset{NESBITT}{\geq}\frac{3}{2}$$

$$\frac{xz}{y^2+yz}+\frac{y^2}{xz+yz}+\frac{yz}{xz+y^2}+\frac{x+2z}{x+z}\geq\frac{3}{2}+\frac{x+2z}{x+z}$$

$$\frac{xz}{y^2+yz}+\frac{y^2}{xz+yz}+\frac{x+2z}{x+z}\geq\frac{3}{2}+1+\frac{z}{x+z}-\frac{yz}{xz+y^2}$$

$$\frac{xz}{y^2+yz}+\frac{y^2}{xz+yz}+\frac{x+2z}{x+z}\geq\frac{5}{2}+\frac{z}{x+z}-\frac{yz}{xz+y^2}$$

Let us prove:

$$\frac{z}{x+z}-\frac{yz}{xz+y^2}\geq 0 (z\leq x)(*)$$

$$\frac{1}{x+y}-\frac{y}{xz+y^2}\geq 0$$

$$y^2-(x+z)y+xz\geq 0$$

y	0	z	x	$+\infty$
$y^2-(x+z)y+xz$	∥	0	0	

Inequality $(*)$ *is true*

1348. If $a,b,c>0$ *and* $abc(a+2)(b+2)(c+2)=27$ then:

$$2^a+2^b+2^c\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$abc(a+2)(b+2)(c+2)=27$$

$$\frac{(a+b+c)^3}{27}\cdot\frac{\big((a+2)+(b+2)+(c+2)\big)^3}{27}\overset{Am-Gm}{\geq}\geq abc(a+2)(b+2)(c+2)=27\ or$$

$$x^3(x+6)^3\overset{x=a+b+c}{\geq}(27)^3\ or$$

$$x(x+6)\geq 27\ or\ x^2+6x\geq 27\ or$$

$$(x+3)^2\geq 36\ or\ x+3\geq 6$$

$$or\ x\geq 3\ (1)$$

$$2^a+2^b+2^c\overset{Am-Gm}{\geq}3\left(2^{a+b+c}\right)^{\frac{1}{3}}\overset{(1)}{\geq}3(2^3)^{\frac{1}{3}}=6$$

$$Equality\ for\ a=b=c=1$$

1349. If $x,y,z>0,\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1$ then:

$$xyz(x+y+z)(x+1)(y+1)(z+1)\geq 1296$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}=1\ \ (1)$$

$$or\ x+y+z+2=xyz\ (2)$$

$$Now\ [(x+1)+(y+1)+(z+1)]\left[\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right]\overset{C-S}{\geq}9\ or$$
$$(x+y+z+3)\overset{(1)}{\geq}9\ or\ x+y+z\geq 6\ (2)$$

$$Now\ from(2)\ xyz\geq 6+2=8\ \ (3)$$

$$and\ xy+yz+zx\overset{AM-GM}{\geq}3(xyz)^{\frac{2}{3}}\overset{(3)}{\geq}3.(8)^{\frac{2}{3}}=12\ \ (4)then$$

$$xyz(x+y+z)(x+1)(y+1)(z+1)=$$

$$= xyz(x+y+z)(1+x+y+z+xy+yz+zx+xyz) \overset{(2)\&(3)\&(4)}{\geq}$$
$$\geq 8.6(1+6+12+8) = 1296$$

$$Equality\ for\ x=y=z=2$$

1350. If $a,b,c>0$, then prove that :

$$\sum_{cyc}\frac{\sqrt{b(b^2-bc+c^2)}+\sqrt{a(c^2-ca+a^2)}}{\sqrt{bc}+\sqrt{ac}}\geq\sqrt{3(a+b+c)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle
$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$
$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{\sqrt{b(b^2-bc+c^2)}+\sqrt{a(c^2-ca+a^2)}}{\sqrt{bc}+\sqrt{ac}}+$$
$$\frac{\sqrt{b(a^2-ab+b^2)}+\sqrt{c(c^2-ca+a^2)}}{\sqrt{ab}+\sqrt{ac}}+\frac{\sqrt{a(a^2-ab+b^2)}+\sqrt{c(b^2-bc+c^2)}}{\sqrt{ab}+\sqrt{bc}}$$

$$=\frac{\sqrt{\frac{b^2-bc+c^2}{a}}+\sqrt{\frac{c^2-ca+a^2}{b}}}{\sqrt{\frac{c}{a}}+\sqrt{\frac{c}{b}}}+\frac{\sqrt{\frac{a^2-ab+b^2}{c}}+\sqrt{\frac{c^2-ca+a^2}{b}}}{\sqrt{\frac{a}{c}}+\sqrt{\frac{a}{b}}}$$
$$+\frac{\sqrt{\frac{a^2-ab+b^2}{c}}+\sqrt{\frac{b^2-bc+c^2}{a}}}{\sqrt{\frac{b}{c}}+\sqrt{\frac{b}{a}}}$$

$$=\frac{\frac{1}{\sqrt{a}}}{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}}\cdot\left(\sqrt{\frac{c^2-ca+a^2}{b}}+\sqrt{\frac{a^2-ab+b^2}{c}}\right)$$
$$+\frac{\frac{1}{\sqrt{b}}}{\frac{1}{\sqrt{c}}+\frac{1}{\sqrt{a}}}\cdot\left(\sqrt{\frac{a^2-ab+b^2}{c}}+\sqrt{\frac{b^2-bc+c^2}{a}}\right)$$
$$+\frac{\frac{1}{\sqrt{c}}}{\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}}\cdot\left(\sqrt{\frac{b^2-bc+c^2}{a}}+\sqrt{\frac{c^2-ca+a^2}{b}}\right)$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{\sqrt{a}},y=\frac{1}{\sqrt{b}},z=\frac{1}{\sqrt{c}},A=\sqrt{\frac{b^2-bc+c^2}{a}},B=\sqrt{\frac{c^2-ca+a^2}{b}},\right.$$
$$\left.C=\sqrt{\frac{a^2-ab+b^2}{c}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$
$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}$$
$$=\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{b^2-bc+c^2}{a}}.\sqrt{\frac{c^2-ca+a^2}{b}}\right)}\overset{A-G}{\geq}\sqrt{3}.\sqrt{\sum_{cyc}\left(\sqrt{\frac{bc+}{a}}.\sqrt{\frac{ca}{b}}\right)}$$
$$=\sqrt{3(a+b+c)}\ \forall\ a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1351. $\forall\, a,b,c>0, prove\ that:$

$$\sum_{cyc}\frac{\sqrt{\frac{b^n+c^n}{a^n+b^n}}+\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\frac{a+c}{a+b}\cdot\sqrt{b^n+c^n}+\frac{b+c}{a+b}\cdot\sqrt{a^n+c^n}}\geq\frac{3\sqrt{3}}{\sqrt{2(a^n+b^n+c^n)}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\sqrt{\frac{b^n+c^n}{a^n+b^n}}+\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\frac{a+c}{a+b}\cdot\sqrt{b^n+c^n}+\frac{b+c}{a+b}\cdot\sqrt{a^n+c^n}}+\frac{\sqrt{\frac{a^n+b^n}{b^n+c^n}}+\sqrt{\frac{a^n+c^n}{b^n+c^n}}}{\frac{a+c}{b+c}\cdot\sqrt{a^n+b^n}+\frac{a+b}{b+c}\cdot\sqrt{a^n+c^n}}$$

$$+\frac{\sqrt{\frac{a^n+b^n}{a^n+c^n}}+\sqrt{\frac{b^n+c^n}{a^n+c^n}}}{\frac{b+c}{a+c}\cdot\sqrt{a^n+b^n}+\frac{a+b}{a+c}\cdot\sqrt{b^n+c^n}}$$

$$=\frac{\left(\frac{a+b}{\sqrt{a^n+b^n}}\right)\left(\sqrt{b^n+c^n}+\sqrt{c^n+a^n}\right)}{\frac{b+c}{\sqrt{b^n+c^n}}\cdot\sqrt{(b^n+c^n)(c^n+a^n)}+\frac{c+a}{\sqrt{c^n+a^n}}\cdot\sqrt{(c^n+a^n)(b^n+c^n)}}$$

$$+\frac{\left(\frac{b+c}{\sqrt{b^n+c^n}}\right)\left(\sqrt{c^n+a^n}+\sqrt{a^n+b^n}\right)}{\frac{c+a}{\sqrt{c^n+a^n}}\cdot\sqrt{(c^n+a^n)(a^n+b^n)}+\frac{a+b}{\sqrt{a^n+b^n}}\cdot\sqrt{(a^n+b^n)(c^n+a^n)}}$$

$$+\frac{\left(\frac{c+a}{\sqrt{c^n+a^n}}\right)\left(\sqrt{a^n+b^n}+\sqrt{b^n+c^n}\right)}{\frac{a+b}{\sqrt{a^n+b^n}}\cdot\sqrt{(a^n+b^n)(b^n+c^n)}+\frac{b+c}{\sqrt{b^n+c^n}}\cdot\sqrt{(b^n+c^n)(a^n+b^n)}}$$

$$=\frac{\frac{a+b}{\sqrt{a^n+b^n}}}{\frac{b+c}{\sqrt{b^n+c^n}}+\frac{c+a}{\sqrt{c^n+a^n}}}\cdot\left(\frac{1}{\sqrt{b^n+c^n}}+\frac{1}{\sqrt{c^n+a^n}}\right)$$

$$+\frac{\frac{b+c}{\sqrt{b^n+c^n}}}{\frac{c+a}{\sqrt{c^n+a^n}}+\frac{a+b}{\sqrt{a^n+b^n}}}\cdot\left(\frac{1}{\sqrt{c^n+a^n}}+\frac{1}{\sqrt{a^n+b^n}}\right)$$

$$+\frac{\frac{c+a}{\sqrt{c^n+a^n}}}{\frac{a+b}{\sqrt{a^n+b^n}}+\frac{b+c}{\sqrt{b^n+c^n}}}\cdot\left(\frac{1}{\sqrt{a^n+b^n}}+\frac{1}{\sqrt{b^n+c^n}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{a+b}{\sqrt{a^n+b^n}},y=\frac{b+c}{\sqrt{b^n+c^n}},z=\frac{c+a}{\sqrt{c^n+a^n}},\right.$$
$$\left.A=\frac{1}{\sqrt{a^n+b^n}},B=\frac{1}{\sqrt{b^n+c^n}},C=\frac{1}{\sqrt{c^n+a^n}}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\frac{1}{\sqrt{(a^n+b^n)(b^n+c^n)}}}$$

$$\overset{\text{Bergstrom}}{\geq}\sqrt{\frac{27}{\sum_{cyc}\sqrt{(a^n+b^n)(b^n+c^n)}}}\overset{\text{CBS}}{\geq}\sqrt{\frac{27}{\sqrt{\left(\sum_{cyc}(a^n+b^n)\right)\left(\sum_{cyc}(b^n+c^n)\right)}}}$$

$$=\frac{3\sqrt{3}}{\sqrt{2(a^n+b^n+c^n)}}\ \forall a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1352. If $a,b,c>0$ and $\lambda\geq 0$ with $n\in\mathbb{N}$, then :

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}=\sum_{cyc}\left(\frac{1}{2a}+\frac{1}{2b}-\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}\right)$$

$$=\sum_{cyc}\frac{(b+a)\left(a^{n+1}+\lambda b^{n+1}\right)-2ab(a^n+\lambda b^n)}{2ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\sum_{cyc}\frac{\lambda b^{n+2}-\lambda ab^{n+1}+a^{n+2}-a^{n+1}b}{2ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\sum_{cyc}\frac{\lambda b^{n+1}(b-a)-a^{n+1}(b-a)}{2ab(a^{n+1}+\lambda b^{n+1})}=\sum_{cyc}\frac{(b-a)\left(\lambda b^{n+1}-a^{n+1}\right)}{2ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\sum_{cyc}\frac{(b-a)\left(\lambda b^{n+1}+a^{n+1}-2a^{n+1}\right)}{2ab(a^{n+1}+\lambda b^{n+1})}=\sum_{cyc}\frac{b-a}{2ab}-\sum_{cyc}\frac{a^{n+1}(b-a)}{ab(a^{n+1}+\lambda b^{n+1})}$$

$$=\frac{1}{2abc}.\sum_{cyc}c(b-a)-\sum_{cyc}\frac{a^{n+1}c\left(b^{n+1}+\lambda c^{n+1}\right)\left(c^{n+1}+\lambda a^{n+1}\right)(b-a)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$=0-\frac{\sum_{cyc}\left(a^{n+1}c\left(b^{n+1}c^{n+1}+\lambda\left(a^{n+1}b^{n+1}+c^8\right)+\lambda^2c^{n+1}a^{n+1}\right)(b-a)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$=-\frac{a^{n+1}b^{n+1}c^{n+1}\sum_{cyc}c(b-a)+\lambda.\sum_{cyc}\left(a^{n+1}c\left(a^{n+1}b^{n+1}+c^{2n+2}\right)(b-a)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$-\frac{\lambda^2.\sum_{cyc}\left(a^{2n+2}c^{n+2}(b-a)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$$=\frac{\lambda\left(\sum_{cyc}a^{2n+3}b^{n+2}-abc\left(\sum_{cyc}a^{2n+1}b^{n+1}\right)\right)+\lambda^2\left(\sum_{cyc}a^{n+1}b^{2n+3}-abc\left(\sum_{cyc}a^{n+1}b^{2n+1}\right)\right)}{abc(a^{n+1}+\lambda b^{n+1})(b^{n+1}+\lambda c^{n+1})(c^{n+1}+\lambda a^{n+1})}$$

$\because\lambda\geq 0\ \therefore$ in order to prove : $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}$, it sufficess to prove :

$$\boxed{\sum_{cyc}a^{2n+3}b^{n+2}\overset{(*)}{\geq}abc\left(\sum_{cyc}a^{2n+1}b^{n+1}\right)}\ \text{and}\ \boxed{\sum_{cyc}a^{n+1}b^{2n+3}\overset{(\bullet)}{\geq}abc\left(\sum_{cyc}a^{n+1}b^{2n+1}\right)}$$

Let's prove $(*)$ for $n=0,1,2$ and indeed we have, $\sum_{cyc}a^3b^2=abc\sum_{cyc}\frac{a^2b}{c}$

$$= abc\sum_{cyc}\frac{a^2b^2}{bc} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab} = abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow \sum_{cyc}a^3b^2 \geq abc\left(\sum_{cyc}ab\right) \rightarrow (1)$$

$$and, \sum_{cyc}a^5b^3 = abc\sum_{cyc}\frac{a^4b^2}{c} = abc\sum_{cyc}\frac{a^6b^4}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc}a^3b^2\right)^2}{abc\left(\sum_{cyc}ab\right)}$$

$$\overset{via\ (1)}{\geq} \frac{abc\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^3b^2\right)}{\sum_{cyc}ab} \Rightarrow \sum_{cyc}a^5b^3 \geq abc\left(\sum_{cyc}a^3b^2\right) \rightarrow (2)$$

$$and, \sum_{cyc}a^7b^4 = abc\sum_{cyc}\frac{a^6b^3}{c} = abc\sum_{cyc}\frac{a^{10}b^6}{a^4b^3c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc}a^5b^3\right)^2}{abc\left(\sum_{cyc}a^3b^2\right)}$$

$$\overset{via\ (2)}{\geq} \frac{abc\left(\sum_{cyc}a^3b^2\right)\left(\sum_{cyc}a^5b^3\right)}{\sum_{cyc}a^3b^2} \Rightarrow \sum_{cyc}a^7b^4 \geq abc\left(\sum_{cyc}a^5b^3\right) \rightarrow (3)$$

$\therefore (1),(2),(3) \Rightarrow \boxed{(*)\text{ is true for } n = 0,1 \text{ and } 2}$ and we now assume $(*)$ is true for $0,1,2,\dots,n$ and we shall subsequently prove that : $(*)$ is true for $n \equiv n+1$

$$\text{For } n \equiv n+1, (*) \Leftrightarrow \sum_{cyc}a^{2n+5}b^{n+3} \overset{(**)}{\geq} abc\left(\sum_{cyc}a^{2n+3}b^{n+2}\right) \text{ and } \sum_{cyc}a^{2n+5}b^{n+3} =$$

$$abc\sum_{cyc}\frac{a^{2n+4}b^{n+2}}{c} = abc\sum_{cyc}\frac{a^{4n+6}b^{2n+4}}{ca^{2n+2}b^{n+2}} \overset{\text{Bergstrom}}{\geq} \frac{abc\left(\sum_{cyc}a^{2n+3}b^{n+2}\right)^2}{abc\left(\sum_{cyc}a^{2n+1}b^{n+1}\right)}$$

$$\overset{via\ (*),our\ assumption}{\geq} \frac{abc\left(\sum_{cyc}a^{2n+1}b^{n+1}\right)\left(\sum_{cyc}a^{2n+3}b^{n+2}\right)}{\sum_{cyc}a^{2n+1}b^{n+1}} = abc\left(\sum_{cyc}a^{2n+3}b^{n+2}\right)$$

$\Rightarrow (**)$ is true whenever $(*)$ is true

$\Rightarrow$ via principle of mathematical induction, $\boxed{(*)\text{ is true } \forall\, n \in \mathbb{N}}$

Let's prove $(\bullet)$ for $n = 0,1,2$ and indeed we have, $\sum_{cyc}a^2b^3 = abc\sum_{cyc}\frac{ab^2}{c}$

$$= abc\sum_{cyc}\frac{a^2b^2}{ca} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc}ab\right)^2}{\sum_{cyc}ab} = abc\left(\sum_{cyc}ab\right)$$

$$\Rightarrow \sum_{cyc}a^2b^3 \geq abc\left(\sum_{cyc}ab\right) \rightarrow (4)$$

$$and, \sum_{cyc}a^3b^5 = abc\sum_{cyc}\frac{a^2b^4}{c} = abc\sum_{cyc}\frac{a^4b^6}{a^2b^2c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc}a^2b^3\right)^2}{abc\left(\sum_{cyc}ab\right)}$$

$$\overset{\text{via (4)}}{\geq} \frac{abc\left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^3\right)}{\sum_{cyc} ab} \Rightarrow \sum_{cyc} a^3b^5 \geq abc\left(\sum_{cyc} a^2b^3\right) \rightarrow (5)$$

$$and, \sum_{cyc} a^4b^7 = abc\sum_{cyc}\frac{a^3b^6}{c} = abc\sum_{cyc}\frac{a^6b^{10}}{a^3b^4c} \overset{\text{Bergstrom}}{\geq} abc.\frac{\left(\sum_{cyc} a^3b^5\right)^2}{abc\left(\sum_{cyc} a^2b^3\right)}$$

$$\overset{\text{via (5)}}{\geq} \frac{abc\left(\sum_{cyc} a^2b^3\right)\left(\sum_{cyc} a^3b^5\right)}{\sum_{cyc} a^2b^3} \Rightarrow \sum_{cyc} a^4b^7 \geq abc\left(\sum_{cyc} a^3b^5\right) \rightarrow (6)$$

$\therefore \boxed{(4),(5),(6) \Rightarrow (\bullet) \text{ is true for } n = 0,1 \text{ and } 2}$ and we now assume $(\bullet)$ is true for $0,1,2,\ldots,n$ and we shall subsequently prove that : $(\bullet)$ is true for $n \equiv n+1$

$$\text{For } n \equiv n+1, (*) \Leftrightarrow \sum_{cyc} a^{n+3}b^{2n+5} \overset{(\bullet\bullet)}{\geq} abc\left(\sum_{cyc} a^{n+2}b^{2n+3}\right) \text{ and } \sum_{cyc} a^{n+3}b^{2n+5} =$$

$$abc\sum_{cyc}\frac{a^{n+2}b^{2n+4}}{c} = abc\sum_{cyc}\frac{a^{2n+4}b^{4n+6}}{ca^{n+2}b^{2n+2}} \overset{\text{Bergstrom}}{\geq} \frac{abc\left(\sum_{cyc} a^{n+2}b^{2n+3}\right)^2}{abc\left(\sum_{cyc} a^{n+1}b^{2n+1}\right)}$$

$$\overset{\text{via } (\bullet), \text{our assumption}}{\geq} \frac{abc\left(\sum_{cyc} a^{n+1}b^{2n+1}\right)\left(\sum_{cyc} a^{n+2}b^{2n+3}\right)}{\sum_{cyc} a^{n+1}b^{2n+1}} = abc\left(\sum_{cyc} a^{n+2}b^{2n+3}\right)$$

$\Rightarrow (\bullet\bullet)$ is true whenever $(\bullet)$ is true

$\Rightarrow$ via principle of mathematical induction, $\boxed{(\bullet) \text{ is true } \forall\, n \in \mathbb{N}}$

$\therefore (*)$ and $(\bullet)$ are both true $\forall\, n \in \mathbb{N} \Rightarrow \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \sum_{cyc}\frac{a^n+\lambda b^n}{a^{n+1}+\lambda b^{n+1}}$

$\forall\, a,b,c > 0 \,|\, \lambda \geq 0$ and $n \in \mathbb{N}$, $''='' $ iff $a = b = c$ (QED)

1353. If $a,b,c > 0$ and $\lambda \geq 1$, then :

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)} \geq \frac{3}{\lambda+1}+\frac{1}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)}-\frac{3}{\lambda+1}-\frac{1}{4}$$

$$=\sum_{cyc}\frac{a^2}{a^2+\lambda ab}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)}-\frac{3}{\lambda+1}-\frac{1}{4} \overset{\text{Bergstrom}}{\geq}$$

$$\frac{\sum_{cyc} a^2+2\sum_{cyc} ab}{\sum_{cyc} a^2+\lambda\sum_{cyc} ab}-\frac{3}{\lambda+1}+\frac{\sum_{cyc} a^2}{4\sum_{cyc} ab}-\frac{1}{4}=\frac{x+2y}{x+\lambda y}-\frac{3}{\lambda+1}+\frac{x-y}{4y}$$

$$\left(x=\sum_{cyc}a^2\ \text{and}\ y=\sum_{cyc}ab\right)=\frac{x+\lambda x+2y+2\lambda y-3x-3\lambda y}{(\lambda+1)(x+\lambda y)}+\frac{x-y}{4y}$$

$$=\frac{\lambda(x-y)-2(x-y)}{(\lambda+1)(x+\lambda y)}+\frac{x-y}{4y}=(x-y)\left(\frac{\lambda-2}{(\lambda+1)(x+\lambda y)}+\frac{1}{4y}\right)$$

$$=\frac{(x-y)(4\lambda y-8y+\lambda x+x+\lambda^2 y+\lambda y)}{4y(\lambda+1)(x+\lambda y)}\overset{\lambda\geq 1\ \text{and}\ x-y\geq 0}{\geq}\frac{(x-y)(4y-8y+x+x+y+y)}{4y(\lambda+1)(x+\lambda y)}$$

$$=\frac{2(x-y)^2}{4y(\lambda+1)(x+\lambda y)}\geq 0\ \therefore\ \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{a^2+b^2+c^2}{4(ab+bc+ca)}\geq$$

$$\frac{3}{\lambda+1}+\frac{1}{4}\ \forall a,b,c>0\,|\,\lambda\geq 1,''=''\ \text{iff}\ a=b=c\ (\text{QED})$$

1354. If $a,b,c>0$, $abc=1, n\in N$ then:

$$\sum_{cyc}\frac{(a+b)^{n+2}}{a^2+3}\geq 3\cdot 2^n$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ will\ show: 4\left(\sum a\right)^2\geq 3\left(\sum a^2+9\right)\ (1)$$

$$proof: 4\left(\sum a\right)^2\geq 3\left(\sum a^2+9\right)\ or\ \sum a^2+8\sum ab\geq 27$$

$$\sum a^2+8\sum ab\overset{Am-Gm}{\geq} 3(abc)^{\frac{2}{3}}+24(abc)^{\frac{2}{3}}=27\ (as\ abc=1)$$

$$\left(\sum\frac{(a+b)^{n+2}}{a^2+3}\right)\left(\sum(a^2+3)\right)(1+1+1)^n\overset{Holder}{\geq}(2a+2b+2c)^{n+2}$$

$$or\left(\sum\frac{(a+b)^{n+2}}{a^2+3}\right)\geq\frac{(2a+2b+2c)^{n+2}}{(\sum(a^2+3))3^n}=\frac{2^n(a+b+c)^n}{3^n}\cdot\frac{4(a+b+c)^2}{a^2+b^2+c^2+9}\overset{Am-Gm}{\geq}$$

$$\geq\frac{2^n3^n(abc)^{\frac{n}{3}}}{3^n}\frac{4(a+b+c)^2}{a^2+b^2+c^2+9}\overset{(1)\&abc=1}{\geq}2^n\cdot 3$$

$$Equality\ for\ a=b=c=1$$

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ROMANIAN MATHEMATICAL MAGAZINE
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1355. If $a,b,c>0$ and $n\in\mathbb{N}$ with $n\geq 2$, then prove that :

$$\sum_{cyc}\frac{(a+c).\sqrt{\frac{b^n+c^n}{a^n+b^n}}+(b+c).\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\sqrt{b^n+c^n}+\sqrt{a^n+c^n}}\geq\frac{3\sqrt{6}.\sqrt[3]{abc}}{\sqrt{a^n+b^n+c^n}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{(a+c).\sqrt{\frac{b^n+c^n}{a^n+b^n}}+(b+c).\sqrt{\frac{a^n+c^n}{a^n+b^n}}}{\sqrt{b^n+c^n}+\sqrt{a^n+c^n}}+\frac{(a+c).\sqrt{\frac{a^n+b^n}{b^n+c^n}}+(a+b).\sqrt{\frac{a^n+c^n}{b^n+c^n}}}{\sqrt{a^n+b^n}+\sqrt{a^n+c^n}}$$

$$+\frac{(b+c).\sqrt{\frac{a^n+b^n}{a^n+c^n}}+(a+b).\sqrt{\frac{b^n+c^n}{a^n+c^n}}}{\sqrt{a^n+b^n}+\sqrt{b^n+c^n}}$$

$$=\frac{\left(\frac{a+c}{\sqrt{a^n+c^n}}\right)}{\frac{\sqrt{a^n+b^n}}{1}}+\frac{\left(\frac{b+c}{\sqrt{b^n+c^n}}\right)}{\frac{\sqrt{a^n+b^n}}{1}}+\frac{\left(\frac{a+c}{\sqrt{a^n+c^n}}\right)}{\frac{\sqrt{b^n+c^n}}{1}}+\frac{\left(\frac{a+b}{\sqrt{a^n+b^n}}\right)}{\frac{\sqrt{b^n+c^n}}{1}}+\frac{\left(\frac{b+c}{\sqrt{b^n+c^n}}\right)}{\frac{\sqrt{a^n+c^n}}{1}}+\frac{\left(\frac{a+b}{\sqrt{a^n+b^n}}\right)}{\frac{\sqrt{a^n+c^n}}{1}}$$

$$=\frac{\frac{1}{\sqrt{a^n+b^n}}}{\frac{1}{\sqrt{b^n+c^n}}+\frac{1}{\sqrt{c^n+a^n}}}.\left(\frac{b+c}{\sqrt{b^n+c^n}}+\frac{c+a}{\sqrt{c^n+a^n}}\right)+$$

$$\frac{\frac{1}{\sqrt{b^n+c^n}}}{\frac{1}{\sqrt{c^n+a^n}}+\frac{1}{\sqrt{a^n+b^n}}}\cdot\left(\frac{c+a}{\sqrt{c^n+a^n}}+\frac{a+b}{\sqrt{a^n+b^n}}\right)+$$

$$\frac{\frac{1}{\sqrt{c^n+a^n}}}{\frac{1}{\sqrt{a^n+b^n}}+\frac{1}{\sqrt{b^n+c^n}}}\cdot\left(\frac{a+b}{\sqrt{a^n+b^n}}+\frac{b+c}{\sqrt{b^n+c^n}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{\sqrt{a^n+b^n}},y=\frac{1}{\sqrt{b^n+c^n}},z=\frac{1}{\sqrt{c^n+a^n}},\right.$$
$$\left.A=\frac{a+b}{\sqrt{a^n+b^n}},B=\frac{b+c}{\sqrt{b^n+c^n}},C=\frac{c+a}{\sqrt{c^n+a^n}}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{a+b}{\sqrt{a^n+b^n}}.\frac{b+c}{\sqrt{b^n+c^n}}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt[6]{\frac{(a+b)^2(b+c)^2(c+a)^2}{(a^n+b^n)(b^n+c^n)(c^n+a^n)}}\overset{\text{Cesaro and A-G}}{\geq}3.\frac{\sqrt[6]{64a^2b^2c^2}}{\sqrt{\frac{\sum_{cyc}(b^n+c^n)}{3}}}=\frac{6\sqrt{3}.\sqrt[3]{abc}}{\sqrt{2(a^n+b^n+c^n)}}$$

$$=\frac{3\sqrt{6}.\sqrt[3]{abc}}{\sqrt{a^n+b^n+c^n}}\ \forall a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1356. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)\geq\frac{3\sqrt{3}}{\sqrt{a^2+b^2+c^2}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$(a+b+c)^2\overset{CBS}{\leq}3(a^2+b^2+c^2)\ (1)$$

$$\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)\overset{AM-GM}{\geq}\frac{2}{a+b}\frac{\sqrt{a^2+b^2}}{\sqrt[4]{(b^2+c^2)(a^2+c^2)}},$$

similarly

$$\frac{1}{b+c}\left(\sqrt{\frac{b^2+c^2}{a^2+c^2}}+\sqrt{\frac{b^2+c^2}{a^2+b^2}}\right)\geq\frac{2}{c+b}\frac{\sqrt{c^2+b^2}}{\sqrt[4]{(b^2+a^2)(a^2+c^2)}},$$

$$\text{and}$$

$$\frac{1}{a+c}\left(\sqrt{\frac{a^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{b^2+c^2}}\right)\geq\frac{2}{a+c}\frac{\sqrt{a^2+c^2}}{\sqrt[4]{(b^2+c^2)(a^2+b^2)}}$$

$$\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)+\frac{1}{b+c}\left(\sqrt{\frac{b^2+c^2}{a^2+c^2}}+\sqrt{\frac{b^2+c^2}{a^2+b^2}}\right)$$

$$+\frac{1}{a+c}\left(\sqrt{\frac{a^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{b^2+c^2}}\right)=$$

$$\sum\frac{1}{a+b}\left(\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+b^2}{a^2+c^2}}\right)\geq\sum\frac{2}{a+b}\frac{\sqrt{a^2+b^2}}{\sqrt[4]{(b^2+c^2)(a^2+c^2)}}\overset{am-gm}{\geq}$$

$$6\left(\frac{1}{\prod(a+b)}\right)^{\frac{1}{3}}\overset{am-gm}{\geq}\frac{18}{2\sum a}\overset{(1)}{\geq}\frac{9}{\left(3(a^2+b^2+c^2)\right)^{\frac{1}{2}}}=\frac{3\sqrt{3}}{\sqrt{a^2+b^2+c^2}}$$

Equality for a=b=c

1357. If $a,b,c>0$, then prove that :

$$\sum_{cyc}\frac{\frac{a+c}{a+b}\cdot\sqrt{b^2+c^2}+\frac{b+c}{a+b}\cdot\sqrt{a^2+c^2}}{b+c+a+c}\geq\frac{3}{\sqrt{2}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\rightarrow(1)$$

$$\text{Now}, \forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{\frac{a+c}{a+b}\cdot\sqrt{b^2+c^2}+\frac{b+c}{a+b}\cdot\sqrt{a^2+c^2}}{b+c+a+c}+\frac{\frac{a+c}{b+c}\cdot\sqrt{a^2+b^2}+\frac{a+b}{b+c}\cdot\sqrt{a^2+c^2}}{a+b+a+c}$$
$$+\frac{\frac{b+c}{a+c}\cdot\sqrt{a^2+b^2}+\frac{a+b}{a+c}\cdot\sqrt{b^2+c^2}}{a+b+b+c}$$
$$=\frac{\frac{1}{(a+b)(b+c)}\cdot\sqrt{b^2+c^2}+\frac{1}{(a+b)(a+c)}\cdot\sqrt{a^2+c^2}}{\frac{1}{a+c}+\frac{1}{b+c}}+$$
$$\frac{\frac{a+c}{(b+c)(a+b)}\cdot\sqrt{a^2+b^2}+\frac{a+b}{(b+c)(a+c)}\cdot\sqrt{a^2+c^2}}{\frac{1}{a+c}+\frac{1}{a+b}}+$$
$$\frac{\frac{1}{(a+c)(a+b)}\cdot\sqrt{a^2+b^2}+\frac{1}{(a+c)(b+c)}\cdot\sqrt{b^2+c^2}}{\frac{1}{b+c}+\frac{1}{a+b}}$$
$$=\frac{\frac{1}{a+b}}{\frac{1}{b+c}+\frac{1}{c+a}}\cdot\left(\frac{\sqrt{b^2+c^2}}{b+c}+\frac{\sqrt{c^2+a^2}}{c+a}\right)+\frac{\frac{1}{b+c}}{\frac{1}{c+a}+\frac{1}{a+b}}\cdot\left(\frac{\sqrt{c^2+a^2}}{c+a}+\frac{\sqrt{a^2+b^2}}{a+b}\right)$$
$$+\frac{\frac{1}{c+a}}{\frac{1}{a+b}+\frac{1}{b+c}}\cdot\left(\frac{\sqrt{a^2+b^2}}{a+b}+\frac{\sqrt{b^2+c^2}}{b+c}\right)$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(x=\frac{1}{a+b},y=\frac{1}{b+c},z=\frac{1}{c+a},A=\frac{\sqrt{a^2+b^2}}{a+b},B=\frac{\sqrt{b^2+c^2}}{b+c},C=\frac{\sqrt{c^2+a^2}}{c+a}\right)$$
$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{a^2+b^2}}{a+b}\cdot\frac{\sqrt{b^2+c^2}}{b+c}\right)}$$
$$\geq\sqrt{3\sum_{cyc}\left(\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right)}=\sqrt{\frac{9}{2}}\therefore\frac{\frac{a+c}{a+b}\cdot\sqrt{b^2+c^2}+\frac{b+c}{a+b}\cdot\sqrt{a^2+c^2}}{b+c+a+c}+$$

$$\frac{\frac{a+c}{b+c}\cdot\sqrt{a^2+b^2}+\frac{a+b}{b+c}\cdot\sqrt{a^2+c^2}}{a+b+a+c}+\frac{\frac{b+c}{a+c}\cdot\sqrt{a^2+b^2}+\frac{a+b}{a+c}\cdot\sqrt{b^2+c^2}}{a+b+b+c}\geq\frac{3}{\sqrt{2}}$$

$\forall a,b,c>0,''=''$ iff $a=b=c$ (QED)

1358. $\forall a,b,c>0, prove\ that:$

$$\sum_{cyc}\frac{\sqrt{\frac{b^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{a^2+b^2}}}{\frac{a+c}{a+b}\cdot\sqrt{b^2+c^2}+\frac{b+c}{a+b}\cdot\sqrt{a^2+c^2}}\geq\frac{3\sqrt{3}}{\sqrt{2(a^2+b^2+c^2)}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\sqrt{\frac{b^2+c^2}{a^2+b^2}}+\sqrt{\frac{a^2+c^2}{a^2+b^2}}}{\frac{a+c}{a+b}\cdot\sqrt{b^2+c^2}+\frac{b+c}{a+b}\cdot\sqrt{a^2+c^2}}+\frac{\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\sqrt{\frac{a^2+c^2}{b^2+c^2}}}{\frac{a+c}{b+c}\cdot\sqrt{a^2+b^2}+\frac{a+b}{b+c}\cdot\sqrt{a^2+c^2}}$$

$$+\frac{\sqrt{\frac{a^2+b^2}{a^2+c^2}}+\sqrt{\frac{b^2+c^2}{a^2+c^2}}}{\frac{b+c}{a+c}\cdot\sqrt{a^2+b^2}+\frac{a+b}{a+c}\cdot\sqrt{b^2+c^2}}$$

$$=\frac{\left(\frac{a+b}{\sqrt{a^2+b^2}}\right)\left(\sqrt{b^2+c^2}+\sqrt{c^2+a^2}\right)}{\frac{b+c}{\sqrt{b^2+c^2}}\cdot\sqrt{(b^2+c^2)(c^2+a^2)}+\frac{c+a}{\sqrt{c^2+a^2}}\cdot\sqrt{(c^2+a^2)(b^2+c^2)}}$$

$$+\frac{\left(\frac{b+c}{\sqrt{b^2+c^2}}\right)\left(\sqrt{c^2+a^2}+\sqrt{a^2+b^2}\right)}{\frac{c+a}{\sqrt{c^2+a^2}}\cdot\sqrt{(c^2+a^2)(a^2+b^2)}+\frac{a+b}{\sqrt{a^2+b^2}}\cdot\sqrt{(a^2+b^2)(c^2+a^2)}}$$

$$+\frac{\left(\frac{c+a}{\sqrt{c^2+a^2}}\right)\left(\sqrt{a^2+b^2}+\sqrt{b^2+c^2}\right)}{\frac{a+b}{\sqrt{a^2+b^2}}\cdot\sqrt{(a^2+b^2)(b^2+c^2)}+\frac{b+c}{\sqrt{b^2+c^2}}\cdot\sqrt{(b^2+c^2)(a^2+b^2)}}$$

$$=\frac{\frac{a+b}{\sqrt{a^2+b^2}}}{\frac{b+c}{\sqrt{b^2+c^2}}+\frac{c+a}{\sqrt{c^2+a^2}}}\cdot\left(\frac{1}{\sqrt{b^2+c^2}}+\frac{1}{\sqrt{c^2+a^2}}\right)$$

$$+\frac{\frac{b+c}{\sqrt{b^2+c^2}}}{\frac{c+a}{\sqrt{c^2+a^2}}+\frac{a+b}{\sqrt{a^2+b^2}}}\cdot\left(\frac{1}{\sqrt{c^2+a^2}}+\frac{1}{\sqrt{a^2+b^2}}\right)+$$

$$+\frac{\frac{c+a}{\sqrt{c^2+a^2}}}{\frac{a+b}{\sqrt{a^2+b^2}}+\frac{b+c}{\sqrt{b^2+c^2}}}\cdot\left(\frac{1}{\sqrt{a^2+b^2}}+\frac{1}{\sqrt{b^2+c^2}}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{a+b}{\sqrt{a^2+b^2}},y=\frac{b+c}{\sqrt{b^2+c^2}},z=\frac{c+a}{\sqrt{c^2+a^2}},\right.$$
$$\left.A=\frac{1}{\sqrt{a^2+b^2}},B=\frac{1}{\sqrt{b^2+c^2}},C=\frac{1}{\sqrt{c^2+a^2}}\right)$$

$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F\cdot\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\frac{1}{\sqrt{(a^2+b^2)(b^2+c^2)}}}$$

$$\overset{\text{Bergstrom}}{\geq}\sqrt{\frac{27}{\sum_{cyc}\sqrt{(a^2+b^2)(b^2+c^2)}}}\overset{\text{CBS}}{\geq}\sqrt{\frac{27}{\sqrt{\left(\sum_{cyc}(a^2+b^2)\right)\left(\sum_{cyc}(b^2+c^2)\right)}}}$$

$$=\frac{3\sqrt{3}}{\sqrt{2(a^2+b^2+c^2)}}\ \forall a,b,c>0,''='' \text{ iff } a=b=c\ (QED)$$

1359. If $a,b,c>0$ then:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq\frac{15}{8}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq\frac{15}{8}\ or$$

$$\sum\frac{a^2}{a^2+ab}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq\frac{15}{8}\ or$$

$$\frac{(a+b+c)^2}{a^2+b^2+c^2+ab+bc+ca}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq\frac{15}{8}or$$

$$\frac{\sum a^2+2\sum ab}{a^2+b^2+c^2+ab+bc+ca}+\frac{9(a^2+b^2+c^2)}{8(\sum a^2+2\sum ab)}\geq\frac{15}{8}\quad(1)$$

$$let\ \frac{a^2+b^2+c^2}{ab+bc+ca}=t\geq 1\ \ then\ from(1)we\ get:$$

$$\frac{t+2}{t+1}+\frac{9t}{8(t+2)}\geq\frac{15}{8}\ or\ 8(t+2)^2+9t(t+1)\geq 15(t+1)(t+2)$$

$$or\ 8(t^2+4t+4)+9t^2+9t\geq 15(t^2+3t+2)or$$

$$2(t^2-2t+1)\geq 0\ or\ (t-1)^2\geq 0\ (true)$$

$$Equality\ for\ t=1\ or\ a^2+b^2+c^2=ab+bc+ca\ or\ a=b=c$$

1360. If $a,b,c,d>0\ and\ \frac{1}{\sqrt{a+15}}+\frac{1}{\sqrt{b+15}}+\frac{1}{\sqrt{c+15}}+\frac{1}{\sqrt{d+15}}=1$ then:

$$3^a+3^b+3^c+3^d\geq 12$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$1=\frac{1^{\frac{3}{2}}}{\sqrt{a+15}}+\frac{1^{\frac{3}{2}}}{\sqrt{b+15}}+\frac{1^{\frac{3}{2}}}{\sqrt{c+15}}+\frac{1^{\frac{3}{2}}}{\sqrt{d+15}}\overset{Radon}{\geq}$$

$$\geq\frac{(4)^{\frac{3}{2}}}{(a+b+c+d+60)^{\frac{1}{2}}}$$

$$or(a+b+c+d+60)^{\frac{1}{2}}\geq(4)^{\frac{3}{2}}$$

$$or\ a+b+c+d+60 \geq 64$$

$$or\ a+b+c+d \geq 4\ \ (1)$$

- $3^a+3^b+3^c+3^d \overset{AM-GM}{\geq} 4\left(3^{a+b+c+d}\right)^{\frac{1}{4}} \overset{(1)}{\geq} 4(3^4)^{\frac{1}{4}} = 12$

- $Equality\ holds\ for\ a=b=c=d=1$

1361. If $x,y,z>0$, $\sqrt{xy}+\sqrt{yz}+\sqrt{zx}=1$, $\lambda \geq 0$ then:

$$\sum \frac{x^3}{y(z+\lambda x)} \geq \frac{1}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx}=1$$

$$\sqrt{3(xy+yz+zx)} \geq 1(CBS)$$

$$\sqrt{\frac{3(\sum x)^2}{3}} \geq 1\ or\ x+y+z \geq 1\ \ (1)$$

$$\sum \frac{x^3}{y(z+\lambda x)} \overset{Holder}{\geq} \frac{(x+y+z)^3}{3(\lambda+1)(xy+yz+zx)} \geq \frac{(x+y+z)^3}{\frac{3(\lambda+1)(x+y+z)^2}{3}}$$

$$= \frac{x+y+z}{\lambda+1} \overset{(1)}{\geq} \frac{1}{\lambda+1}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{3}$$

1362. If $a,b,c>0$, $\lambda \geq \frac{9}{8}$ then:

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{(a+b+c)^2} \geq \lambda\frac{1}{3}+\frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ \frac{(a^2+b^2+c^2)}{ab+bc+ca}=t\geq 1$$

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}+\frac{\lambda(a^2+b^2+c^2)}{(a+b+c)^2}=\sum\frac{a^2}{a^2+ab}+\frac{\lambda(a^2+b^2+c^2)}{(a+b+c)^2}\geq$$

$$\geq\frac{(a+b+c)^2}{a^2+b^2.+c^2+ab+bc+ca}+\frac{\lambda(a^2+b^2+c^2)}{(a+b+c)^2}=\frac{t+2}{t+1}+\frac{\lambda t}{t+2}$$

$$We\ \ need\ to\ show:\ \ \frac{t+2}{t+1}+\frac{\lambda t}{t+2}\geq\lambda\frac{1}{3}+\frac{3}{2}$$

$$t^2(4\lambda-3)-3t+6-4\lambda\geq 0\ or\ (t-1)[(4\lambda-3)(t+1)-3]\geq 0$$

$$true\ as\ \ t\geq 1\ and\ (4\lambda-3)(t+1)>\left(\frac{4.9}{8}-3\right)(1+1)=(1.5).3>3$$

$$since\ \lambda\geq\frac{9}{8}.\ Equality\ for\ a=b=c.$$

1363. If $a,b,c>0$ and $\lambda\geq 1$, then :

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq\frac{3}{\lambda+1}+\frac{3}{8}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\ n\geq\frac{9}{8},\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}-\frac{3}{\lambda+1}+\frac{n(a^2+b^2+c^2)}{(a+b+c)^2}-\frac{n}{3}$$

$$=\sum_{cyc}\frac{a^2}{a^2+\lambda ab}-\frac{3}{\lambda+1}+n\left(\frac{\sum_{cyc}a^2}{\left(\sum_{cyc}a\right)^2}-\frac{1}{3}\right)$$

$$\overset{Bergstrom}{\geq}\frac{\sum_{cyc}a^2+2\sum_{cyc}ab}{\sum_{cyc}a^2+\lambda\sum_{cyc}ab}-\frac{3}{\lambda+1}+n\left(\frac{3\sum_{cyc}a^2-\left(\sum_{cyc}a\right)^2}{3\left(\sum_{cyc}a\right)^2}\right)$$

$$\overset{n\geq\frac{9}{8}}{\geq}\frac{x+2y}{x+\lambda y}-\frac{3}{\lambda+1}+\frac{9}{8}\left(\frac{3x-x-2y}{3(x+2y)}\right)\left(x=\sum_{cyc}a^2\ and\ y=\sum_{cyc}ab\right)$$

$$= \frac{x+\lambda x+2y+2\lambda y-3x-3\lambda y}{(\lambda+1)(x+\lambda y)}+\frac{3(x-y)}{4(x+2y)} = \frac{\lambda(x-y)-2(x-y)}{(\lambda+1)(x+\lambda y)}+\frac{3(x-y)}{4(x+2y)}$$

$$= (x-y)\left(\frac{\lambda-2}{(\lambda+1)(x+\lambda y)}+\frac{3}{4(x+2y)}\right)$$

$$= \frac{(x-y)\left(4(\lambda-2)(x+2y)+3(\lambda+1)(x+\lambda y)\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$= \frac{(x-y)\left((3\lambda^2+11\lambda-16)y+7\lambda x-5x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$= \frac{(x-y)\left((3\lambda^2+11\lambda-16)y+2\lambda x+5\lambda x-5x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$\overset{x\geq y}{\geq} \frac{(x-y)\left((3\lambda^2+11\lambda-16)y+2\lambda y+5(\lambda-1)x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)}$$

$$= \frac{(x-y)\left((3\lambda^2+13\lambda-16)y+5(\lambda-1)x\right)}{4(\lambda+1)(x+\lambda y)(x+2y)} \geq 0$$

$$\because \lambda-1\geq 0 \text{ and } 3\lambda^2+13\lambda-16 \overset{\lambda\geq 1}{\geq} 0 \text{ and } x-y\geq 0$$

$$\therefore \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{n(a^2+b^2+c^2)}{(a+b+c)^2}\geq \frac{n}{3}+\frac{3}{\lambda+1}$$

$$\forall a,b,c>0 \,|\, n\geq \frac{9}{8} \text{ and } \lambda\geq 1 \text{ and putting } n=\frac{9}{8}, \text{we get :}$$

$$\frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}-\frac{3}{\lambda+1}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}-\frac{3}{8}\geq 0$$

$$\Rightarrow \frac{a}{a+\lambda b}+\frac{b}{b+\lambda c}+\frac{c}{c+\lambda a}+\frac{9(a^2+b^2+c^2)}{8(a+b+c)^2}\geq \frac{3}{\lambda+1}+\frac{3}{8}$$

$$\forall a,b,c>0 \,|\, \lambda\geq 1, '' = '' \text{ iff } a=b=c \text{ (QED)}$$

1364.

If $a,b,c>0$ *with* $abc=1$ and $\lambda\geq 2, n\geq 0$, then :

$$\sum_{cyc}\frac{a^3+\lambda}{a^3(b+nc)}\geq \frac{3(\lambda+1)}{n+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3+\lambda}{a^3(b+nc)}=\sum_{cyc}\frac{a}{ab+nca}+\lambda.\sum_{cyc}\frac{b^2c^2}{a^3b^3c^2+nc^3a^3b^2}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{\left(\sum_{cyc}\sqrt{a}\right)^2}{(n+1)\left(\sum_{cyc}ab\right)}+\frac{\lambda\left(\sum_{cyc}ab\right)^2}{a^2b^2c^2\left(\sum_{cyc}ab\right)}\overset{abc=1}{=}$$

$$\frac{1}{n+1}.\left(\frac{\sum_{cyc}a+2\sum_{cyc}\sqrt{ab}}{\sum_{cyc}ab}+\lambda\sum_{cyc}ab\right)$$

$$\geq\frac{1}{n+1}.\left(\frac{\sqrt{3\sum_{cyc}ab}+2\sqrt{abc}.\sum_{cyc}\frac{1^{\frac{3}{2}}}{\sqrt{a}}}{\sum_{cyc}ab}+\lambda\sum_{cyc}ab\right)\overset{\text{Radon and }\because abc=1}{\geq}$$

$$\frac{1}{n+1}.\left(\frac{\sqrt{3\sum_{cyc}ab}+2.\frac{\sqrt{27}}{\sqrt{\sum_{cyc}a}}}{\sum_{cyc}ab}+\lambda\sum_{cyc}ab\right)\overset{abc=1}{=}$$

$$\frac{1}{n+1}.\left(\frac{\sqrt{3\sum_{cyc}ab}+18.\frac{1}{\sqrt{3abc\sum_{cyc}a}}}{\sum_{cyc}ab}+\lambda\sum_{cyc}ab\right)$$

$$\geq\frac{1}{n+1}.\left(\frac{\sqrt{3\sum_{cyc}ab}+\frac{18}{\sum_{cyc}ab}}{\sum_{cyc}ab}+\lambda\sum_{cyc}ab\right)$$

$$=\frac{1}{n+1}.\left(\frac{x+\frac{54}{x^2}}{\frac{x^2}{3}}+\lambda.\frac{x^2}{3}\right)\left(x=\sqrt{3\sum_{cyc}ab}\overset{A-G}{\geq}3.\sqrt[6]{a^2b^2c^2}\overset{abc=1}{=}3\right)$$

$$=\frac{1}{n+1}.\left(\frac{9x^3+486+\lambda x^6}{3x^4}\right)\overset{?}{\geq}\frac{3(\lambda+1)}{n+1}\Leftrightarrow\lambda x^4(x^2-9)+9x^3+486-9x^4\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, } x^2-9\geq 0 \text{ and } \lambda\geq 0$$

$$\therefore \text{LHS of } (*)\geq 2x^4(x^2-9)+9x^3+486-9x^4$$

$$=2x^6-27x^4+9x^3+486=$$

$$=(x-3)\left((x-3)(2x^4+12x^3+27x^2+63x+135)+243\right)\geq 0\ (\because x\geq 3)$$

$$\Rightarrow (*)\text{ is true }\therefore\sum_{cyc}\frac{a^3+\lambda}{a^3(b+nc)}\geq\frac{3(\lambda+1)}{n+1}$$

$$\forall a,b,c>0\,|\,abc=1 \text{ and } \lambda\geq 2, n\geq 0,''='' \text{ iff } a=b=c=1\ (\text{QED})$$

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1365. If $a,b>0$ and $n\in\mathbb{N}$ with $n\geq 2$, then :

$$4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4n+2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

WLOG we may assume $a\geq b$ and then : $\frac{a}{b+1}\geq\frac{b}{a+1}$ and

$a^{n-1}\geq b^{n-1}\ (\because n-1\geq 1)\ \therefore 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)\overset{\text{Chebyshev}}{\geq}$

$$2\left(\frac{a}{b+1}+\frac{b}{a+1}\right)(a^{n-1}+b^{n-1})$$

$$\overset{\text{Bernoulli}}{\geq} 2\left(\frac{a}{b+1}+\frac{b}{a+1}\right)(1+(a-1)(n-1)+1+(b-1)(n-1))$$

$$=2\left(\frac{a}{b+1}+\frac{b}{a+1}\right)(2+(n-1)(a+b-2))$$

$$=4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+2(n-1)(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)$$

$$\Rightarrow 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2)\geq$$

$$4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+2(n-1)(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)$$

$$+2(n-1)\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{a}+\frac{1}{b}-4(n-1)-6$$

$$=2(n-1)\left((a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\right)$$

$$+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-6$$

$$\therefore \boxed{\begin{array}{c}4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2)\geq\\ 2(n-1)\left((a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\right)+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-6\end{array}}$$

$$\rightarrow (1)$$

$$\text{Now, } (a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2$$

$$=(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{4}{a+b}+\frac{(a-b)^2}{ab(a+b)}-2$$

$$=(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}-\frac{2}{a+b}\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$=(a+b-2)\left(\left(\frac{a}{b+1}-\frac{1}{a+b}\right)+\left(\frac{b}{a+1}-\frac{1}{a+b}\right)\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$=(a+b-2)\left(\frac{a^2+ab-b-1}{(a+b)(b+1)}+\frac{b^2+ab-a-1}{(a+b)(a+1)}\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$=(a+b-2)\left(\frac{(a-1)(a+1)+b(a-1)}{(a+b)(b+1)}+\frac{(b-1)(b+1)+a(b-1)}{(a+b)(a+1)}\right)$$

$$+\frac{(a-b)^2}{ab(a+b)}=\frac{(a+b-2)(a+b+1)}{a+b}\cdot\left(\frac{a-1}{b+1}+\frac{b-1}{a+1}\right)+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b-2)(a+b+1)}{a+b}\cdot\frac{a^2-1+b^2-1}{(b+1)(a+1)}+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b-2)(a+b+1)((a+b)^2+(a-b)^2-4)}{2(a+b)(a+1)(b+1)}+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b-2)(a+b+1)\left((a+b-2)(a+b+2)+(a-b)^2\right)}{2(a+b)(a+1)(b+1)}+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b+1)(a+b+2)(a+b-2)^2}{2(a+b)(a+1)(b+1)}+\frac{(a+b-2)(a+b+1)(a-b)^2}{2(a+b)(a+1)(b+1)}$$

$$+\frac{(a-b)^2}{ab(a+b)}$$

$$=\frac{(a+b+1)(a+b+2)(a+b-2)^2}{2(a+b)(a+1)(b+1)}+\frac{(a-b)^2}{a+b}\cdot\left(\frac{(a+b-2)(a+b+1)}{2(a+1)(b+1)}+\frac{1}{ab}\right)$$

$$\therefore\ \boxed{\begin{aligned}&(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2=\frac{(a+b+1)(a+b+2)(a+b-2)^2}{2(a+b)(a+1)(b+1)}\\&+\frac{(a-b)^2}{2ab(a+b)(a+1)(b+1)}\cdot\left(ab(a+b-2)(a+b+1)+2(a+b+1+ab)\right)\end{aligned}}$$

$$\rightarrow (2)$$

$$\text{Now, } ab(a+b-2)(a+b+1)+2(a+b+1+ab)=$$

$$\frac{ab(a+b-2)(a+b+1)}{2}+2(a+b+1)+\frac{ab(a+b-2)(a+b+1)}{2}+2ab$$

$$=\frac{a+b+1}{2}\cdot(ab(a+b)-2ab+4)+\frac{ab}{2}\cdot((a+b)^2-(a+b)-2+4)\overset{A-G}{\geq}$$

$$\frac{a+b+1}{2}\cdot\left(2ab\cdot\sqrt{ab}-2ab+4\right)+\frac{ab}{2}\cdot\left(\left(a+b-\frac{1}{2}\right)^2+\frac{7}{4}\right)$$

$$=(a+b+1)(t^3-t^2+2)+\frac{ab}{2}\cdot\left(\left(a+b-\frac{1}{2}\right)^2+\frac{7}{4}\right)\ (t=\sqrt{ab})$$

$$=(a+b+1)(t+1)((t-1)^2+1)+\frac{ab}{2}\cdot\left(\left(a+b-\frac{1}{2}\right)^2+\frac{7}{4}\right)>0$$

$$\Rightarrow \boxed{ab(a+b-2)(a+b+1)+2(a+b+1+ab)>0}\rightarrow (3)\ \therefore (2),(3)\Rightarrow$$

$$(a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\geq 0\rightarrow(4)$$

$$\therefore (1),(4)\ and\ (n-1)\geq 1\Rightarrow 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2)$$

$$\geq 2\left((a+b-2)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-2\right)+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+\frac{1}{a}+\frac{1}{b}-6$$

$$=2(a+b)\left(\frac{a}{b+1}+\frac{b}{a+1}\right)-4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)+3\left(\frac{1}{a}+\frac{1}{b}\right)-10$$

$$+4\left(\frac{a}{b+1}+\frac{b}{a+1}\right)\overset{\text{Bergstrom}}{\geq}$$

$$2(a+b)\left(\frac{(a+b)^2}{2ab+a+b}\right)+\frac{12}{a+b}-10\geq 2(a+b)\left(\frac{(a+b)^2}{\frac{(a+b)^2}{2}+a+b}\right)+\frac{12}{a+b}-10$$

$$\overset{?}{\geq}0\Leftrightarrow 2m.\frac{m^2}{\frac{m^2}{2}+m}+\frac{12}{m}-10\overset{?}{\geq}0\ (m=a+b)\Leftrightarrow 2m^3-5m^2-4m-12\overset{?}{\geq}0$$

$$\Leftrightarrow (2m+3)(m-2)^2\geq 0\rightarrow true$$

$$\therefore 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)-(4n+2)\geq 0$$

$$\Rightarrow 4\left(\frac{a^n}{b+1}+\frac{b^n}{a+1}\right)+(2n-1)\left(\frac{1}{a}+\frac{1}{b}\right)\geq 4n+2$$

$$\forall a,b>0\ and\ n\in\mathbb{N}\ with\ n\geq 2,''='' \ iff\ a=b=1\ (QED)$$

1366. If $a,b,c\geq 0,\ a^2+b^2+c^2=1$ then:

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}\geq\frac{4}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}=\sum\frac{1}{a^3+2}=\frac{1}{2}\sum\frac{2}{a^3+2}=$$

$$=\frac{1}{2}\sum\left(1-\frac{a^3}{a^3+1+1}\right)\overset{AM-GM}{\geq}\frac{3}{2}-\frac{1}{2}\sum\frac{a^3}{3a}=$$

$$=\frac{3}{2}-\frac{1}{6}\sum a^2=\frac{3}{2}-\frac{1}{6}=\frac{4}{3}$$

$$Equality\ for\ (a,b,c)=(1,0,0)or(0,1,0)or(0,0,1)$$

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1367. If $a,b,c>0$ then:

$$\frac{c\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}}{b(a+b)}+\frac{c\sqrt{a^2+b^2}+a\sqrt{c^2+a^2}}{c(b+c)}+\frac{b\sqrt{a^2+b^2}+a\sqrt{b^2+c^2}}{a(c+a)}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\forall x,y>0\,,\frac{\left(x^2+y^2\right)}{2}\overset{CBS}{\geq}\left(\frac{x+y}{2}\right)^2\quad(1)$$

$$\frac{c\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}}{b(a+b)}\overset{(1)}{\geq}\frac{c\sqrt{\frac{(b+c)^2}{2}}+b\sqrt{\frac{(c+a)^2}{2}}}{b(a+b)}=\frac{1}{\sqrt{2}}\frac{\left(c(b+c)+b(c+a)\right)}{b(a+b)}=$$

$$=\frac{1}{\sqrt{2}}\left[\frac{c}{b}\frac{b+c}{a+b}+\frac{c+a}{a+b}\right]$$

Analogous:

$$\frac{c\sqrt{a^2+b^2}+a\sqrt{c^2+a^2}}{c(b+c)}\geq\frac{1}{\sqrt{2}}\left[\frac{a}{c}\frac{c+a}{b+c}+\frac{a+b}{b+c}\right]$$

$$\frac{b\sqrt{a^2+b^2}+a\sqrt{b^2+c^2}}{a(c+a)}\geq\frac{1}{\sqrt{2}}\left[\frac{b}{a}\frac{a+b}{c+a}+\frac{b+c}{c+a}\right]$$

By adding:

$$\frac{c\sqrt{b^2+c^2}+b\sqrt{c^2+a^2}}{b(a+b)}+\frac{c\sqrt{a^2+b^2}+a\sqrt{c^2+a^2}}{c(b+c)}+\frac{b\sqrt{a^2+b^2}+a\sqrt{b^2+c^2}}{a(c+a)}\geq$$

$$\geq\frac{1}{\sqrt{2}}\left(\left[\frac{c}{b}\frac{b+c}{a+b}+\frac{c+a}{a+b}\right]+\left[\frac{a}{c}\frac{c+a}{b+c}+\frac{a+b}{b+c}\right]+\left[\frac{b}{a}\frac{a+b}{c+a}+\frac{b+c}{c+a}\right]\right)\overset{AM-GM}{\geq}$$

$$\geq\frac{6}{\sqrt{2}}\left(\left[\frac{c}{b}\frac{b+c}{a+b}\cdot\frac{c+a}{a+b}\right]\cdot\left[\frac{a}{c}\frac{c+a}{b+c}\cdot\frac{a+b}{b+c}\right]\cdot\left[\frac{b}{a}\frac{a+b}{c+a}\cdot\frac{b+c}{c+a}\right]\right)^{\frac{1}{6}}=$$

$$=\frac{6}{\sqrt{2}}\left(\frac{(abc(a+b)^2(b+c)^2(c+a)^2)}{abc(a+b)^2(b+c)^2(c+a)^2}\right)^{\frac{1}{6}}=3\sqrt{2}$$

$$Equality\ \ holds\ for\ a=b=c$$

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1368. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{c.\sqrt{b+c}+b.\sqrt{c+a}}{b(a+b)}+\frac{c.\sqrt{a+b}+a.\sqrt{c+a}}{c(b+c)}+\frac{b.\sqrt{a+b}+a.\sqrt{b+c}}{a(c+a)}\geq 3\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{c.\sqrt{b+c}+b.\sqrt{c+a}}{b(a+b)}+\frac{c.\sqrt{a+b}+a.\sqrt{c+a}}{c(b+c)}+\frac{b.\sqrt{a+b}+a.\sqrt{b+c}}{a(c+a)}$$

$$\overset{abc=1}{=}\frac{\frac{\sqrt{b+c}}{ab}+\frac{\sqrt{c+a}}{ca}}{b(a+b)}+\frac{\frac{\sqrt{a+b}}{ab}+\frac{\sqrt{c+a}}{bc}}{c(b+c)}+\frac{\frac{\sqrt{a+b}}{ca}+\frac{\sqrt{b+c}}{bc}}{a(c+a)}$$

$\frac{1}{a}=bc,\ \frac{1}{b}=ca,\ \frac{1}{c}=ab$

$$=\frac{bc\left(\frac{\sqrt{b+c}}{b}+\frac{\sqrt{c+a}}{c}\right)}{b(a+b)}+\frac{ca\left(\frac{\sqrt{a+b}}{a}+\frac{\sqrt{c+a}}{c}\right)}{c(b+c)}+\frac{ab\left(\frac{\sqrt{a+b}}{a}+\frac{\sqrt{b+c}}{b}\right)}{a(c+a)}$$

$$=\frac{c}{a+b}.\left(\frac{\sqrt{b+c}}{b}+\frac{\sqrt{c+a}}{c}\right)+\frac{a}{b+c}.\left(\frac{\sqrt{c+a}}{c}+\frac{\sqrt{a+b}}{a}\right)$$

$$+\frac{b}{c+a}.\left(\frac{\sqrt{a+b}}{a}+\frac{\sqrt{b+c}}{b}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=c,y=a,z=b,A=\frac{\sqrt{a+b}}{a},B=\frac{\sqrt{b+c}}{b},C=\frac{\sqrt{c+a}}{c}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{a+b}}{a}.\frac{\sqrt{b+c}}{b}\right)}\overset{A-G}{\geq}$$

$$3.\sqrt{\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{a^2b^2c^2}}}\overset{abc=1}{=}3.\sqrt[6]{\frac{(a+b)(b+c)(c+a)}{abc}}\overset{\text{Cesaro}}{\geq}3.\sqrt[6]{8}=3\sqrt{2}$$

$$\therefore\frac{c.\sqrt{b+c}+b.\sqrt{c+a}}{b(a+b)}+\frac{c.\sqrt{a+b}+a.\sqrt{c+a}}{c(b+c)}+\frac{b.\sqrt{a+b}+a.\sqrt{b+c}}{a(c+a)}\geq 3\sqrt{2}$$

$\forall a,b,c>0$ and $abc=1$, "=" iff $a=b=c=1$ (QED)

1369. If $a,b,c>0$ and $a^5+b^5+c^5=3$, then prove that :

$$\frac{a^6}{bc}+\frac{b^6}{ca}+\frac{c^6}{ab}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3=\sum_{cyc}a^5\overset{\text{Holder}}{\geq}\frac{1}{81}\left(\sum_{cyc}a\right)^5\Rightarrow\sum_{cyc}a\leq 3\Rightarrow\sum_{cyc}ab\leq\frac{1}{3}\left(\sum_{cyc}a\right)^2\leq\frac{9}{3}$$

$$\therefore\sum_{cyc}ab\leq 3\rightarrow(1)$$

Now, WLOG we may assume $a\geq b\geq c$ and then : $a^6\geq b^6\geq c^6$ and

$$\frac{1}{bc}\geq\frac{1}{ca}\geq\frac{1}{ab}\therefore\frac{a^6}{bc}+\frac{b^6}{ca}+\frac{c^6}{ab}\overset{\text{Chebyshev}}{\geq}\frac{1}{3}\left(\sum_{cyc}a^6\right)\left(\sum_{cyc}\frac{1}{bc}\right)$$

$$\overset{\text{Chebyshev and Bergstrom}}{\geq}\frac{1}{9}\left(\sum_{cyc}a^5\right)\left(\sum_{cyc}a\right).\frac{9}{\sum_{cyc}ab}\geq\frac{3.\sqrt{3\sum_{cyc}ab}}{\sum_{cyc}ab}\left(\because\sum_{cyc}a^5=3\right)=\frac{3\sqrt{3}}{\sqrt{\sum_{cyc}ab}}$$

$$\overset{\text{via (1)}}{\geq}\frac{3\sqrt{3}}{\sqrt{3}}\therefore\frac{a^6}{bc}+\frac{b^6}{ca}+\frac{c^6}{ab}\geq 3\ \forall a,b,c>0\,|\,a^5+b^5+c^5=3,$$

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"=" iff $a=b=c=1$ (QED)

1370. If $a,b,c>0$ and $m,n\in\mathbb{N}$, then prove that :

$$\sum_{cyc}\frac{\left(\frac{b+c}{a+b}\right)^m+\left(\frac{c+a}{a+b}\right)^m}{(b+c)^m.\sqrt{\frac{c^n+a^n}{a^n+b^n}}+(c+a)^m.\sqrt{\frac{b^n+c^n}{a^n+b^n}}}\geq\frac{9}{2^m.(a^m+b^m+c^m)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have : }\frac{\left(\frac{b+c}{a+b}\right)^m+\left(\frac{c+a}{a+b}\right)^m}{(b+c)^m.\sqrt{\frac{c^n+a^n}{a^n+b^n}}+(c+a)^m.\sqrt{\frac{b^n+c^n}{a^n+b^n}}}+$$

$$\frac{\left(\frac{c+a}{b+c}\right)^m+\left(\frac{a+b}{b+c}\right)^m}{(c+a)^m.\sqrt{\frac{a^n+b^n}{b^n+c^n}}+(a+b)^m.\sqrt{\frac{c^n+a^n}{b^n+c^n}}}+\frac{\left(\frac{a+b}{a+c}\right)^m+\left(\frac{b+c}{a+c}\right)^m}{(a+b)^m.\sqrt{\frac{b^n+c^n}{a^n+c^n}}+(b+c)^m.\sqrt{\frac{a^n+b^n}{a^n+c^n}}}$$

$$=\frac{\left(\frac{1}{(a+b)^m(c+a)^m}+\frac{1}{(a+b)^m(b+c)^m}\right).\sqrt{a^n+b^n}}{\frac{1}{(c+a)^m}.\sqrt{c^n+a^n}+\frac{1}{(b+c)^m}.\sqrt{b^n+c^n}}+$$

$$\frac{\left(\frac{1}{(b+c)^m(a+b)^m}+\frac{1}{(b+c)^m(c+a)^m}\right)\cdot\sqrt{b^n+c^n}}{\frac{1}{(a+b)^m}\cdot\sqrt{a^n+b^n}+\frac{1}{(c+a)^m}\cdot\sqrt{c^n+a^n}}$$
$$+\frac{\left(\frac{1}{(c+a)^m(b+c)^m}+\frac{1}{(c+a)^m(a+b)^m}\right)\cdot\sqrt{c^n+a^n}}{\frac{1}{(b+c)^m}\cdot\sqrt{b^n+c^n}+\frac{1}{(a+b)^m}\cdot\sqrt{a^n+b^n}}$$
$$=\frac{\frac{\sqrt{a^n+b^n}}{(a+b)^m}\cdot\left(\frac{1}{(b+c)^m}+\frac{1}{(c+a)^m}\right)}{\frac{\sqrt{b^n+c^n}}{(b+c)^m}+\frac{\sqrt{c^n+a^n}}{(c+a)^m}}+\frac{\frac{\sqrt{b^n+c^n}}{(b+c)^m}\cdot\left(\frac{1}{(c+a)^m}+\frac{1}{(a+b)^m}\right)}{\frac{\sqrt{c^n+a^n}}{(c+a)^m}+\frac{\sqrt{a^n+b^n}}{(a+b)^m}}$$
$$+\frac{\frac{\sqrt{c^n+a^n}}{(c+a)^m}}{\frac{\sqrt{a^n+b^n}}{(a+b)^m}+\frac{\sqrt{b^n+c^n}}{(b+c)^m}}$$
$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(x=\frac{\sqrt{a^n+b^n}}{(a+b)^m},y=\frac{\sqrt{b^n+c^n}}{(b+c)^m},z=\frac{\sqrt{c^n+a^n}}{(c+a)^m},\right.$$
$$\left.A=\frac{1}{(a+b)^m},B=\frac{1}{(b+c)^m},C=\frac{1}{(c+a)^m}\right)$$
$$=\frac{x}{y+z}\cdot\sqrt{B+C}^2+\frac{y}{z+x}\cdot\sqrt{C+A}^2+\frac{z}{x+y}\cdot\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

4F. $$\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}\cdot\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{1}{(a+b)^m}\cdot\frac{1}{(b+c)^m}\right)}$$
$$\overset{\text{Radon}}{\geq}\sqrt{3}\cdot\sqrt{\frac{(1+1+1)^{m+1}}{\left(\sum_{cyc}((a+b)(b+c))\right)^m}}=\sqrt{3}\cdot\sqrt{\frac{3^{m+1}}{\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)^m}}$$
$$\geq\sqrt{3}\cdot\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\left(\sum_{cyc}a\right)^2\right)^m}}\therefore\boxed{LHS\overset{(*)}{\geq}\sqrt{3}\cdot\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\right)^m\cdot\left(\left(\sum_{cyc}a\right)^m\right)^2}}}$$

Case 1 $m=0$ *and then* : $$LHS\overset{\text{via }(*)}{\geq}\sqrt{3}\cdot\sqrt{\frac{3}{\left(\frac{4}{3}\right)^0\cdot\left(\sum_{cyc}a\right)^0}}=3$$
$$=\frac{9}{2^m\cdot(a^m+b^m+c^m)}\ (\because m=0)\therefore LHS\geq\frac{9}{2^m\cdot(a^m+b^m+c^m)}$$

Case 2 $m\in\mathbb{N}^*$ *and then* : $$LHS\overset{\text{via }(*)}{\geq}\sqrt{3}\cdot\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\right)^m\cdot\left(\left(\sum_{cyc}a\right)^m\right)^2}}\overset{\text{Holder}}{\geq}$$

$$\sqrt{3}.\sqrt{\frac{3^{m+1}}{\left(\frac{4}{3}\right)^m.\left(3^{m-1}.\left(\sum_{cyc}a^m\right)\right)^2}}=\sqrt{3}.\sqrt{\frac{\frac{3^{2m+1}}{3^{2m-2}}}{2^{2m}.\left(\sum_{cyc}a^m\right)^2}}=\frac{9}{2^m.(a^m+b^m+c^m)}$$

$$\therefore \text{combining both cases},\frac{\left(\frac{b+c}{a+b}\right)^m+\left(\frac{c+a}{a+b}\right)^m}{(b+c)^m.\sqrt{\frac{c^n+a^n}{a^n+b^n}}+(c+a)^m.\sqrt{\frac{b^n+c^n}{a^n+b^n}}}+$$

$$\frac{\left(\frac{c+a}{b+c}\right)^m+\left(\frac{a+b}{b+c}\right)^m}{(c+a)^m.\sqrt{\frac{a^n+b^n}{b^n+c^n}}+(a+b)^m.\sqrt{\frac{c^n+a^n}{b^n+c^n}}}+\frac{\left(\frac{a+b}{a+c}\right)^m+\left(\frac{b+c}{a+c}\right)^m}{(a+b)^m.\sqrt{\frac{b^n+c^n}{a^n+c^n}}+(b+c)^m.\sqrt{\frac{a^n+b^n}{a^n+c^n}}}$$

$$\geq\frac{9}{2^m.(a^m+b^m+c^m)}\ \forall a,b,c>0 \text{ and } m,n\in\mathbb{N},''='' \text{ iff } a=b=c\ (\text{QED})$$

1371. If $a,b,c>0$, *then prove that* :

$$\sum_{cyc}\frac{\frac{b+c}{a+b}+\frac{c+a}{a+b}}{(b+c).\sqrt{\frac{c^2+a^2}{a^2+b^2}}+(c+a).\sqrt{\frac{b^2+c^2}{a^2+b^2}}}\geq\frac{9}{2(a+b+c)}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle

$(\because (A+B)+(B+C)>(C+A)$ and analogs$)\Rightarrow\sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now},\forall x,y,z>0,\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{\frac{b+c}{a+b}+\frac{c+a}{a+b}}{(b+c).\sqrt{\frac{c^2+a^2}{a^2+b^2}}+(c+a).\sqrt{\frac{b^2+c^2}{a^2+b^2}}}+\frac{\frac{c+a}{b+c}+\frac{a+b}{b+c}}{(c+a).\sqrt{\frac{a^2+b^2}{b^2+c^2}}+(a+b).\sqrt{\frac{c^2+a^2}{b^2+c^2}}}$$

$$+\frac{\frac{a+b}{a+c}+\frac{b+c}{a+c}}{(a+b).\sqrt{\frac{b^2+c^2}{a^2+c^2}}+(b+c).\sqrt{\frac{a^2+b^2}{a^2+c^2}}}$$

$$=\frac{\frac{1}{(a+b)(c+a)}+\frac{1}{(a+b)(b+c)}}{\frac{1}{c+a}.\sqrt{\frac{c^2+a^2}{a^2+b^2}}+\frac{1}{b+c}.\sqrt{\frac{b^2+c^2}{a^2+b^2}}}+\frac{\frac{1}{(b+c)(a+b)}+\frac{1}{(b+c)(c+a)}}{\frac{1}{a+b}.\sqrt{\frac{a^2+b^2}{b^2+c^2}}+\frac{1}{c+a}.\sqrt{\frac{c^2+a^2}{b^2+c^2}}}$$

$$+\frac{\frac{1}{(c+a)(b+c)}+\frac{1}{(c+a)(a+b)}}{\frac{1}{b+c}.\sqrt{\frac{b^2+c^2}{a^2+c^2}}+\frac{1}{a+b}.\sqrt{\frac{a^2+b^2}{a^2+c^2}}}$$

$$=\frac{\frac{\sqrt{a^2+b^2}}{a+b}}{\frac{\sqrt{b^2+c^2}}{b+c}+\frac{\sqrt{c^2+a^2}}{c+a}}.\left(\frac{1}{b+c}+\frac{1}{c+a}\right)+\frac{\frac{\sqrt{b^2+c^2}}{b+c}}{\frac{\sqrt{c^2+a^2}}{c+a}+\frac{\sqrt{a^2+b^2}}{a+b}}.\left(\frac{1}{c+a}+\frac{1}{a+b}\right)+$$

$$\frac{\frac{\sqrt{c^2+a^2}}{c+a}}{\frac{\sqrt{a^2+b^2}}{a+b}+\frac{\sqrt{b^2+c^2}}{b+c}}.\left(\frac{1}{a+b}+\frac{1}{b+c}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{\sqrt{a^2+b^2}}{a+b},y=\frac{\sqrt{b^2+c^2}}{b+c},z=\frac{\sqrt{c^2+a^2}}{c+a},A=\frac{1}{a+b},B=\frac{1}{b+c},C=\frac{1}{c+a}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{1}{a+b}.\frac{1}{b+c}\right)}$$

$$\overset{\text{Bergstrom}}{\geq}3\sqrt{3}.\sqrt{\frac{1}{\sum_{cyc}((a+b)(b+c))}}=\frac{3\sqrt{3}}{\sqrt{\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab}}\geq$$

$$\frac{3\sqrt{3}}{\sqrt{\left(\sum_{cyc}a\right)^2+\frac{\left(\sum_{cyc}a\right)^2}{3}}}=\frac{9}{\sqrt{4\left(\sum_{cyc}a\right)^2}}\therefore\sum_{cyc}\frac{\frac{b+c}{a+b}+\frac{c+a}{a+b}}{(b+c).\sqrt{\frac{c^2+a^2}{a^2+b^2}}+(c+a).\sqrt{\frac{b^2+c^2}{a^2+b^2}}}$$

$$\geq\frac{9}{2(a+b+c)}\ \forall\ a,b,c>0,''=''\ \text{iff}\ \ a=b=c\ (\text{QED})$$

1372. If $a,b,c>0$, then prove that :

$$\frac{2\left(a^5+b^5+c^5+a^2+b^2+c^2\right)}{a^3+b^3+c^3}+\frac{9}{(a+b+c)^2}\geq 5$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2\left(a^5+b^5+c^5+a^2+b^2+c^2\right)}{a^3+b^3+c^3}+\frac{9}{(a+b+c)^2}$$

$$=\frac{\sum_{cyc}a^5}{\sum_{cyc}a^3}+\frac{\sum_{cyc}a^5}{\sum_{cyc}a^3}+\frac{\sum_{cyc}a^2}{\sum_{cyc}a^3}+\frac{\sum_{cyc}a^2}{\sum_{cyc}a^3}+\frac{9}{\left(\sum_{cyc}a\right)^2}\overset{A-G}{\geq}5.\sqrt[5]{\frac{9\left(\sum_{cyc}a^5\right)^2\left(\sum_{cyc}a^2\right)^2}{\left(\sum_{cyc}a^3\right)^4\left(\sum_{cyc}a\right)^2}}$$

$$\geq 5.\sqrt[5]{\frac{\left(\sum_{cyc}a^5\right)^2\left(\sum_{cyc}a\right)^4}{\left(\sum_{cyc}a^3\right)^4\left(\sum_{cyc}a\right)^2}}\left(\because\sum_{cyc}a^2\geq\frac{1}{3}\left(\sum_{cyc}a\right)^2\right)=5.\sqrt[5]{\frac{\left(\sum_{cyc}a^5\right)^2\left(\sum_{cyc}a\right)^2}{\left(\sum_{cyc}a^3\right)^4}}$$

$$\geq 5.\sqrt[5]{\frac{\left(\sum_{cyc}a^3\right)^4}{\left(\sum_{cyc}a^3\right)^4}}\left(\because\left(\sum_{cyc}a^5\right)\left(\sum_{cyc}a\right)\overset{\text{Reverse CBS}}{\geq}\left(\sum_{cyc}a^3\right)^2\right)$$

$$\therefore\frac{2\left(a^5+b^5+c^5+a^2+b^2+c^2\right)}{a^3+b^3+c^3}+\frac{9}{(a+b+c)^2}\geq 5$$

$$\forall\ a,b,c>0,''=''\ \ \text{iff}\ a=b=c\ (\text{QED})$$

1373. If $a,b,c>0$ and $n\in\mathbb{N}^*-\{1\}$, then prove that :

$$\sum_{cyc}\frac{\frac{a+c}{a+b}.\sqrt{b^n+c^n}+\frac{b+c}{a+b}.\sqrt{a^n+c^n}}{b+c+a+c}\geq\frac{3}{\sqrt{2}}(abc)^{\frac{n-2}{6}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$\forall A,B,C>0,(A+B),(B+C),(C+A)$ form sides of a triangle $(\because (A+B)+(B+C)>(C+A)$ and analogs$) \Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A}$ form sides of a triangle with area F (say) and $16F^2=$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2=2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB\Rightarrow 4F=2\sqrt{\sum_{cyc}AB}\to(1)$$

$$\text{Now, }\forall x,y,z>0,\ \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\Leftrightarrow\sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)}\underset{(*)}{\overset{?}{\geq}}\frac{3}{4}$$

$$\text{Via Bergstrom, LHS of }(*)\geq\frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)}=\frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\to\text{true}\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\to(2)$$

$$\text{We have}:\frac{\frac{a+c}{a+b}\cdot\sqrt{b^n+c^n}+\frac{b+c}{a+b}\cdot\sqrt{a^n+c^n}}{b+c+a+c}+\frac{\frac{a+c}{b+c}\cdot\sqrt{a^n+b^n}+\frac{a+b}{b+c}\cdot\sqrt{a^n+c^n}}{a+b+a+c}$$
$$+\frac{\frac{b+c}{a+c}\cdot\sqrt{a^n+b^n}+\frac{a+b}{a+c}\cdot\sqrt{b^n+c^n}}{a+b+b+c}$$
$$=\frac{\frac{1}{(a+b)(b+c)}\cdot\sqrt{b^n+c^n}+\frac{1}{(a+b)(a+c)}\cdot\sqrt{a^n+c^n}}{\frac{1}{a+c}+\frac{1}{b+c}}+$$
$$\frac{\frac{a+c}{(b+c)(a+b)}\cdot\sqrt{a^n+b^n}+\frac{a+b}{(b+c)(a+c)}\cdot\sqrt{a^n+c^n}}{\frac{1}{a+c}+\frac{1}{a+b}}+$$
$$\frac{\frac{1}{(a+c)(a+b)}\cdot\sqrt{a^n+b^n}+\frac{1}{(a+c)(b+c)}\cdot\sqrt{b^n+c^n}}{\frac{1}{b+c}+\frac{1}{a+b}}$$
$$=\frac{\frac{1}{a+b}}{\frac{1}{b+c}+\frac{1}{c+a}}\cdot\left(\frac{\sqrt{b^n+c^n}}{b+c}+\frac{\sqrt{c^n+a^n}}{c+a}\right)+\frac{\frac{1}{b+c}}{\frac{1}{c+a}+\frac{1}{a+b}}\cdot\left(\frac{\sqrt{c^n+a^n}}{c+a}+\frac{\sqrt{a^n+b^n}}{a+b}\right)$$
$$+\frac{\frac{1}{c+a}}{\frac{1}{a+b}+\frac{1}{b+c}}\cdot\left(\frac{\sqrt{a^n+b^n}}{a+b}+\frac{\sqrt{b^n+c^n}}{b+c}\right)$$

$$=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$

$$\left(x=\frac{1}{a+b},y=\frac{1}{b+c},z=\frac{1}{c+a},A=\frac{\sqrt{a^n+b^n}}{a+b},B=\frac{\sqrt{b^n+c^n}}{b+c},C=\frac{\sqrt{c^n+a^n}}{c+a}\right)$$

$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}.\sqrt{A+B}^2\overset{\text{Oppenheim}}{\geq}$$

$$4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{a^n+b^n}}{a+b}.\frac{\sqrt{b^n+c^n}}{b+c}\right)}$$

$$\overset{A-G}{\geq}3.\sqrt{\sqrt{\prod_{cyc}\left(\frac{\sqrt{a^n+b^n}}{a+b}.\frac{\sqrt{b^n+c^n}}{b+c}\right)}}\overset{Cesaro}{\geq}3.\sqrt[6]{\frac{(a^n+b^n)(b^n+c^n)(c^n+a^n)}{(a+b)^2(b+c)^2(c+a)^2}}$$

$$\overset{\text{Chebyshev and Reverse CBS}}{\geq}3.\sqrt[6]{\frac{(a^2+b^2)(a^{n-2}+b^{n-2})}{2}.\frac{(b^2+c^2)(b^{n-2}+c^{n-2})}{2}.\frac{(c^2+a^2)(c^{n-2}+a^{n-2})}{2}}{(2a^2+2b^2)(2b^2+2c^2)(2c^2+2a^2)}}\overset{Cesaro}{\geq}$$

$$3.\sqrt[6]{\frac{8(abc)^{n-2}}{64}}$$

$$\therefore\frac{a+c}{a+b}.\frac{\sqrt{b^n+c^n}}{b+c+a+c}+\frac{b+c}{a+b}.\frac{\sqrt{a^n+c^n}}{b+c+a+c}+\frac{a+c}{b+c}.\frac{\sqrt{a^n+b^n}}{a+b+a+c}+\frac{a+b}{b+c}.\frac{\sqrt{a^n+c^n}}{a+b+a+c}+$$

$$+\frac{b+c}{a+c}.\frac{\sqrt{a^n+b^n}}{a+b+b+c}+\frac{a+b}{a+c}.\frac{\sqrt{b^n+c^n}}{a+b+b+c}\geq\frac{3}{\sqrt{2}}(abc)^{\frac{n-2}{6}}\ \forall a,b,c>0\,|\,n\in\mathbb{N}^*-\{1\},$$

$''='' \text{ iff } a=b=c\ (\text{QED})$

1374. If $a,b,c>0,\frac{1}{\sqrt{a+8}}+\frac{1}{\sqrt{b+8}}+\frac{1}{\sqrt{c+8}}=$ then:

$$2^a+2^b+2^c\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$1=\frac{1}{\sqrt{a+8}}+\frac{1}{\sqrt{b+8}}+\frac{1}{\sqrt{c+8}}=$$

$$=\frac{1^{\frac{3}{2}}}{\sqrt{a+8}}+\frac{1^{\frac{3}{2}}}{\sqrt{b+8}}+\frac{1^{\frac{3}{2}}}{\sqrt{c+8}}\overset{Radon}{\geq}\frac{(3)^{\frac{3}{2}}}{(a+b+c+24)^{\frac{1}{2}}}$$

$$(a+b+c+24)^{\frac{1}{2}} \geq 3^{\frac{3}{2}}$$

$$a+b+c+24 \geq 27$$

$$a+b+c \geq 3$$

$$2^a+2^b+2^c \overset{AM-GM}{\geq} 3\left(2^{a+b+c}\right)^{\frac{1}{3}} \geq 3(2^3)^{\frac{1}{3}} = 6$$

$$Equality\ for\ a=b=c=1$$

1375. If $a,b,c>0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 2$ then:

$$\frac{1}{\sqrt{5a^2+2ab+2b^2}}+\frac{1}{\sqrt{5b^2+2bc+2c^2}}+\frac{1}{\sqrt{5c^2+2ca+2a^2}} \leq \frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$5a^2+2ab+2b^2 = 4a^2+2ab+b^2+(a^2+b^2) \overset{AM-GM}{\geq}$$

$$\geq 4a^2+4ab+b^2 = (2a+b)^2$$

$$LHS \leq \sum \frac{1}{2a+b} = \sum \frac{1}{a+a+b} \overset{AM-HM}{\leq}$$

$$\leq \frac{1}{9}\sum\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = \frac{3}{9}\sum\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq \frac{2}{3}$$

Equality holds for: $a=b=c=\frac{3}{2}$.

1376. If $x,y,z>0$ then:

$$\frac{2x}{x^6+y^4}+\frac{2y}{y^6+z^4}+\frac{2z}{z^6+x^4} \leq \frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{2x}{x^6+y^4}+\frac{2y}{y^6+z^4}+\frac{2z}{z^6+x^4} \overset{AM-GM}{\leq} \sum \frac{2x}{2x^3y^2} = \sum \frac{1}{x^2y^2} = \frac{x^2+y^2+z^2}{x^2y^2z^2} =$$

$$=\frac{x^2y^2z^2(x^2+y^2+z^2)}{x^4y^4z^4}=\frac{\sum(x^2y^2)(x^2z^2)}{x^4y^4z^4}\overset{(\sum ab)\le(\sum a^2)\forall\, a,b,c>0}{\le}$$

$$\le\frac{x^4y^4+y^4z^4+z^4x^4}{x^4y^4z^4}=\frac{1}{x^4}+\frac{1}{y^4}+\frac{1}{z^4}$$

Equality holds for $x=y=z=1$.

1377. If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a^3+5}{a^3(b+c)}+\frac{b^3+5}{b^3(c+a)}+\frac{c^3+5}{c^3(a+b)}\ge 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3+5}{a^3(b+c)}\overset{abc=1}{=}\sum_{cyc}\frac{1}{b+c}+5\sum_{cyc}\frac{b^3c^3}{b+c}\overset{\text{Bergstrom and Holder}}{\ge}\frac{9}{2\sum_{cyc}a}+\frac{5\left(\sum_{cyc}ab\right)^3}{6\sum_{cyc}a}$$

$$\ge\frac{9}{2\sum_{cyc}a}+\frac{15abc\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)}{6\sum_{cyc}a}\overset{abc=1}{=}\frac{9}{2\sum_{cyc}a}+\frac{5\left(\sum_{cyc}ab\right)}{2}$$

$$\ge\frac{9}{2\sum_{cyc}a}+\frac{5.\sqrt{3abc\left(\sum_{cyc}a\right)}}{2}\overset{abc=1}{=}\frac{9}{2\left(\frac{t^2}{3}\right)}+\frac{5t}{2}\left(t=\sqrt{3\sum_{cyc}a}\right)\overset{?}{\ge}9$$

$$\Leftrightarrow 5t^3-18t^2+27\overset{?}{\ge}0\Leftrightarrow(t-3)\big((t-3)(5t+12)+27\big)\overset{?}{\ge}0\to\text{true}$$

$$\because t=\sqrt{3\sum_{cyc}a}\overset{A-G}{\ge}\sqrt{9.\sqrt[3]{abc}}\overset{abc=1}{=}3\therefore\frac{a^3+5}{a^3(b+c)}+\frac{b^3+5}{b^3(c+a)}+\frac{c^3+5}{c^3(a+b)}\ge 9$$

$\forall\, a,b,c>0\,|\,abc=1,''='' \text{ iff } a=b=c=1$ (QED)

1378. If $a,b,c>0$ and $abc=\frac{1}{6}$, then prove that :

$$\frac{a^2}{(1+a)^2}+\frac{4b^2}{(1+2b)^2}+\frac{36c^3}{(1+3c)^3}\ge\frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We consider } f(t)=\frac{4t^3}{3(t+1)^3}-\frac{t^2}{(t+1)^2}\ \forall\ t>0;\ then:$$

$$f'(t)=\frac{2t(t-1)}{(t+1)^4}\geq 0\ \forall\ t\geq 1\ and\ \leq 0\ \forall\ t\leq 1\ \therefore f(t)\ is\uparrow on\ [1,\infty)\ and\ f(t)\ is\downarrow on$$

$$(0,1]\ \therefore f(t)\ is\ minimum\ at\ t=1\Rightarrow f(t)\geq f(1)=-\frac{1}{12}$$

$$\therefore \frac{4t^3}{3(t+1)^3}-\frac{t^2}{(t+1)^2}\geq -\frac{1}{12}\ \forall\ t>0\to(1)$$

$$\text{Let } a=x, 2b=y \text{ and } 3c=z; \text{ then : } xyz=1\ \left(\because abc=\frac{1}{6}\right)$$

$$and\ \frac{a^2}{(1+a)^2}+\frac{4b^2}{(1+2b)^2}+\frac{36c^3}{(1+3c)^3}=\frac{x^2}{(x+1)^2}+\frac{y^2}{(1+y)^2}+\frac{4z^3}{3(1+z)^3}$$

$$=\frac{x^2}{(x+1)^2}+\frac{y^2}{(1+y)^2}+\frac{z^2}{(z+1)^2}+\frac{4z^3}{3(1+z)^3}-\frac{z^2}{(z+1)^2}\overset{via\ (1)}{\geq}$$

$$\frac{x^2}{(x+1)^2}+\frac{y^2}{(1+y)^2}+\frac{z^2}{(z+1)^2}-\frac{1}{12}\overset{?}{\geq}\frac{2}{3}\Leftrightarrow\frac{x^2}{(x+1)^2}+\frac{y^2}{(1+y)^2}+\frac{z^2}{(z+1)^2}\boxed{\overset{?}{\geq}}\frac{3}{4}$$

$$(*)$$

$$\text{Now, } \frac{x^2}{(x+1)^2}+\frac{y^2}{(y+1)^2}+\frac{z^2}{(z+1)^2}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x^2+2\sum_{cyc}x+3}$$

$$=\frac{\sum_{cyc}x^2+2\sum_{cyc}xy}{\sum_{cyc}x^2+2\sum_{cyc}x+3}\overset{?}{\geq}\frac{3}{4}\Leftrightarrow 4\sum_{cyc}x^2+8\sum_{cyc}xy\overset{?}{\geq}3\sum_{cyc}x^2+6\sum_{cyc}x+9$$

$$\Leftrightarrow\sum_{cyc}x^2+8\sum_{cyc}xy-6\sum_{cyc}x-9\boxed{\overset{?}{\geq}}0$$

$$(**)$$

$$\text{Now, LHS of } (**)\geq\frac{1}{3}\left(\sum_{cyc}x\right)^2+8.\sqrt{3xyz\sum_{cyc}x}-6\sum_{cyc}x-9$$

$$\overset{xyz=1}{\geq}\frac{1}{3}\left(\sum_{cyc}x\right)^2+8.\sqrt{3\sum_{cyc}x}-6\sum_{cyc}x-9$$

$$=\frac{u^4}{9}+24u-18.\frac{u^2}{3}-27\left(u=\sqrt{3\sum_{cyc}x}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow u^4-54u^2+216u-243\overset{?}{\geq}0\Leftrightarrow(u+9)(u-3)^3\overset{?}{\geq}0\to true\ \because u=\sqrt{3\sum_{cyc}x}$$

$$\overset{A-G}{\geq}\sqrt{9.\sqrt[3]{xyz}}\overset{xyz=1}{=}3\Rightarrow(**)\Rightarrow(*)\ is\ true\ \therefore\frac{a^2}{(1+a)^2}+\frac{4b^2}{(1+2b)^2}+\frac{36c^3}{(1+3c)^3}$$

$$\geq \frac{2}{3}\ \forall\ a,b,c>0\ \big|\ abc=\frac{1}{6},''='' \text{ iff } a=1,b=\frac{1}{2},c=\frac{1}{3}\ (\text{QED})$$

1379. If $a,b,c>0$, then prove that :

$$\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{4c^3}{3(c+a)^3}\geq\frac{2}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4c^3}{3(c+a)^3}-\frac{c^2}{(c+a)^2}=\frac{4t^3}{3(t+1)^3}-\frac{t^2}{(t+1)^2}\left(t=\frac{c}{a}\right)\text{ and we consider}$$

$$f(t)=\frac{4t^3}{3(t+1)^3}-\frac{t^2}{(t+1)^2}\ \forall\ t>0;\ then:$$

$$f'(t)=\frac{2t(t-1)}{(t+1)^4}\geq 0\ \forall\ t\geq 1\ and\ \leq 0\ \forall\ t\leq 1$$

$$\therefore f(t)\text{ is }\uparrow\text{ on }[1,\infty)\text{ and }f(t)\text{ is }\downarrow\text{ on }(0,1]\therefore f(t)\text{ is minimum at }t=1\Rightarrow f(t)\geq f(1)$$

$$=-\frac{1}{12}\therefore\frac{4c^3}{3(c+a)^3}-\frac{c^2}{(c+a)^2}\geq-\frac{1}{12}\to(1)$$

$$\text{Now},\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}=\frac{x^2}{(x+1)^2}+\frac{y^2}{(y+1)^2}+\frac{z^2}{(z+1)^2}$$

$$\left(x=\frac{a}{b},y=\frac{b}{c},z=\frac{c}{a}\right)\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x^2+2\sum_{cyc}x+3}=\frac{\sum_{cyc}x^2+2\sum_{cyc}xy}{\sum_{cyc}x^2+2\sum_{cyc}x+3}\overset{?}{\geq}\frac{3}{4}$$

$$\Leftrightarrow 4\sum_{cyc}x^2+8\sum_{cyc}xy\overset{?}{\geq}3\sum_{cyc}x^2+6\sum_{cyc}x+9$$

$$\Leftrightarrow\sum_{cyc}x^2+8\sum_{cyc}xy-6\sum_{cyc}x-9\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, LHS of }(*)\geq\frac{1}{3}\left(\sum_{cyc}x\right)^2+8.\sqrt{3xyz\sum_{cyc}x}-6\sum_{cyc}x-9$$

$$\overset{xyz=1}{\geq}\frac{1}{3}\left(\sum_{cyc}x\right)^2+8.\sqrt{3\sum_{cyc}x}-6\sum_{cyc}x-9$$

$$=\frac{u^4}{9}+24u-18.\frac{u^2}{3}-27\left(u=\sqrt{3\sum_{cyc}x}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow u^4-54u^2+216u-243\overset{?}{\geq}0\Leftrightarrow(u+9)(u-3)^3\overset{?}{\geq}0\to\text{true}\because u=\sqrt{3\sum_{cyc}x}$$

$$=\sqrt{3\sum_{cyc}\frac{a}{b}}\overset{A-G}{\geq}3\Rightarrow(*)\text{ is true}\Rightarrow\frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{c^2}{(c+a)^2}\geq\frac{3}{4}\to(2)$$

$$\therefore (1)+(2) \Rightarrow \frac{a^2}{(a+b)^2}+\frac{b^2}{(b+c)^2}+\frac{4c^3}{3(c+a)^3} \geq \frac{3}{4}-\frac{1}{12}=\frac{2}{3}$$

$$\forall a,b,c>0, '' = '' \text{ iff } a=b=c \text{ (QED)}$$

1380. If $a,b,c \in [1,2]$, then prove that :

$$a^2+b^2+c^2+3\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\forall a,b,c>0, \sum_{cyc} a^2+3\sqrt[3]{a^2b^2c^2} \overset{?}{\geq} 2\sum_{cyc} ab$$

$$\Leftrightarrow \sum_{cyc} a^2+3\sqrt[3]{a^2b^2c^2} \overset{?}{\geq} 2\sum_{cyc} ab$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^3+27a^2b^2c^2+9\left(\sum_{cyc} a^2\right)^2.\sqrt[3]{a^2b^2c^2}$$

$$+27\left(\sum_{cyc} a^2\right).\sqrt[3]{a^4b^4c^4} \boxed{\overset{?}{\geq}_{(*)}} 8\left(\sum_{cyc} ab\right)^3$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s,R,r$ (say);

$$\text{so } 2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \rightarrow (1) \Rightarrow a=s-x, b=s-y, c=s-z$$

$$\therefore abc=r^2s \rightarrow (2) \text{and such substitutions} \Rightarrow \sum_{cyc} ab=\sum_{cyc}(s-x)(s-y) \Rightarrow$$

$$\sum_{cyc} ab=4Rr+r^2 \rightarrow (3), \sum_{cyc} a^2=\left(\sum_{cyc} a\right)^2-2\sum_{cyc} ab \overset{\text{via } (1),(3)}{=} s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} a^2=s^2-8Rr-2r^2 \rightarrow (4)$$

$$\text{Via } (1),(2) \text{ and } (4), \text{LHS of } (*) \geq$$

$$(s^2-8Rr-2r^2)^3+27r^4s^2+9(s^2-8Rr-2r^2)^2.\sqrt[3]{r^4s^2}$$

$$+27(s^2-8Rr-2r^2).\sqrt[3]{r^8s^4} \overset{\text{Mitrinovic}}{\geq}$$

$$(s^2-8Rr-2r^2)^3+27r^4s^2+9(s^2-8Rr-2r^2)^2.\sqrt[3]{r^4.27r^2}$$

$$+27(s^2-8Rr-2r^2).\sqrt[3]{r^8.729r^4}$$

$$= (s^2-8Rr-2r^2)^3+27r^4s^2+27r^2(s^2-8Rr-2r^2)^2+243r^4(s^2-8Rr-2r^2)$$

$$\overset{?}{\geq} 8\left(\sum_{cyc} ab\right)^3 \overset{via\ (3)}{=} 8(4Rr+r^2)^3$$

$$\Leftrightarrow s^6-(24Rr-21r^2)s^4+r^2(192R^2-336Rr+174r^2)s^2$$

$$-r^3(1024R^3-960R^2r+1272Rr^2+394r^3)\boxed{\overset{?}{\geq}}\ 0\ and$$

$$\because (s^2-16Rr+5r^2)^3 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \geq (s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (8R+2r)s^4-r(192R^2-48Rr-33r^2)s^2$$

$$+r^2(1024R^3-960R^2r-24Rr^2-173r^3)\boxed{\overset{(***)}{\geq}}\ 0\ and\ \because$$

$$(8R+2r)(s^2-16Rr+5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (8R+2r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (64R^2+32Rr+13r^2)s^2\boxed{\overset{(****)}{\geq}}r(1024R^3+192R^2r-96Rr^2+223r^3)$$

$$Finally, (64R^2+32Rr+13r^2)s^2 \overset{Gerretsen}{\geq} (64R^2+32Rr+13r^2)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(1024R^3+192R^2r-96Rr^2+223r^3) \Leftrightarrow 144r^3(R-2r) \overset{?}{\geq} 0 \rightarrow true\ via\ Euler$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)\ is\ true$$

$$\therefore \forall a,b,c>0, \sum_{cyc} a^2+3\sqrt[3]{a^2b^2c^2} \geq 2\sum_{cyc} ab$$

$$\therefore a^2+b^2+c^2+3\sqrt[3]{a^2b^2c^2} \geq 2(ab+bc+ca)\ \forall a,b,c \in [1,2],$$

$$'' =''\ \text{iff } a=b=c\ (QED)$$

1381. If $a,b,c>0$ and $a+b+c=3$, then :

$$\sum_{cyc} \frac{bc}{\sqrt[4]{2a^4+14}} \leq \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt[4]{2a^4+14}} \overset{?}{\leq} \frac{9-a}{16} \Leftrightarrow \frac{1}{2a^4+14} \overset{?}{\leq} \frac{(9-a)^4}{65536}$$

$$(\because a,b,c>0 \wedge a+b+c=3 \Rightarrow 9>3>a) \Leftrightarrow (2a^4+14)(9-a)^4 \overset{?}{\geq} 65536$$

$$\Leftrightarrow a^8-36a^7+486a^6-2916a^5+6568a^4-252a^3$$

$$+3402a^2-20412a+13159 \overset{?}{\geq} 0$$

$$\Leftrightarrow (a-1)^2\left(a^6-34a^5+417a^4-2048a^3+2055a^2+5906a+13159\right)\overset{?}{\geq}0$$

$$\Leftrightarrow (a-1)^2((a-8)^2(a^4-18a^3+65a^2+144a+199)+423-126a)\overset{?}{\geq}0$$

$$\Leftrightarrow (a-1)^2\left((a-8)^2(a^2(a-5)(a-13)+144a+199)+45+126(3-a)\right)\overset{?}{\geq}0$$

$\rightarrow$ true $\because 0<a<3\Rightarrow (a-5),(a-13)<0\Rightarrow a^2(a-5)(a-13)>0$ and

$$126(3-a)>0\ \therefore \frac{1}{\sqrt[4]{2a^4+14}}\leq\frac{9-a}{16}\ \text{and analogs}$$

$$\Rightarrow\sum_{cyc}\frac{bc}{\sqrt[4]{2a^4+14}}\leq\sum_{cyc}\frac{bc(9-a)}{16}\overset{?}{\leq}\frac{3}{2}\Leftrightarrow 3\sum_{cyc}ab-abc\overset{?}{\leq}8$$

$$\overset{a+b+c=3}{\Leftrightarrow}\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-abc\overset{?}{\leq}\frac{8}{27}\left(\sum_{cyc}a\right)^3$$

$$\Leftrightarrow 8\sum_{cyc}a^3\underset{(*)}{\overset{?}{\geq}}6abc+3\sum_{cyc}a^2b+3\sum_{cyc}ab^2$$

Now, Schur + AM − GM $\Rightarrow 6\sum_{cyc}a^3\geq 3\sum_{cyc}a^2b+3\sum_{cyc}ab^2\rightarrow(1)$ and AM − GM $\Rightarrow$

$$2\sum_{cyc}a^3\geq 6abc\rightarrow(2)\ \therefore (1)+(2)\Rightarrow(*)\ \text{is true}$$

$$\therefore\sum_{cyc}\frac{bc}{\sqrt[4]{2a^4+14}}\leq\frac{3}{2}\ \forall a,b,c>0\,|\,a+b+c=3,''='' \text{ iff } a=b=1\ (\text{QED})$$

1382. *If* $x,y,z,t>0$, $x+y+z+t=1$ *then:*

$$\sum\frac{1}{x^2+x}\geq\frac{64}{5}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

Lemma:

$$\forall\, m>0,\frac{1}{m^2+m}\geq\frac{(176-384m)}{25},$$

Proof:

$$\frac{1}{m^2+m}\geq\frac{(176-384m)}{25}\ or$$

$$25\geq -384m^3-384\,m^2+176m^2+176m\ \ or$$

$$384m^3+208m^2-176m+25\geq 0\ or$$

$$6(64m^3)+13(16m^2)-44(4m)+25\geq 0\ or$$

$$6(4m)^3+13(4m)^2-44(4m)+25\geq 0\ or$$

$$6n^3+13n^2-44n+25\overset{4m=n>0}{\geq}0\ or$$

$$(n-1)^2(6n+25)\geq 0\ true$$

$$\sum\frac{1}{x^2+x}\overset{lemma}{\geq}\sum\frac{(176-384x)}{25}=\frac{(176).4-384(x+y+z+t)}{25}=$$

$$=\frac{704-384}{25}(as\ x+y+z+t=1)=\frac{320}{25}=\frac{64}{5}$$

$$Equality\ for\ x=y=z=t=\frac{1}{4}$$

1383. *If* $a,b,c>0$ *then*

$$\sum\frac{\sqrt{ab}}{a+b}+\frac{1}{4}\sum\frac{a+b}{c}\geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{\sqrt{ab}}{a+b}+\frac{1}{4}\sum\frac{a+b}{c}=$$

$$=\frac{\sqrt{ab}}{a+b}+\frac{\sqrt{bc}}{b+c}+\frac{\sqrt{ca}}{c+a}+\frac{1}{4}\frac{(a+b)}{c}+\frac{1}{4}\frac{(b+c)}{a}+\frac{1}{4}\frac{c+a}{b}\overset{Am-Gm}{\geq}$$

$$\geq 6\left(\frac{\sqrt{ab}}{a+b}\cdot\frac{\sqrt{bc}}{b+c}\cdot\frac{\sqrt{ca}}{c+a}\cdot\frac{1}{4}\frac{(a+b)}{c}\cdot\frac{1}{4}\frac{(b+c)}{a}\cdot\frac{1}{4}\frac{c+a}{b}\right)^{\frac{1}{6}}=$$

$$=6\left(\frac{abc(a+b)(b+c](c+a)}{64abc(a+b)(b+c)(c+a)}\right)^{\frac{1}{6}}=6.\left(\frac{1}{64}\right)^{\frac{1}{6}}=\frac{6}{2}=3$$

$$Equality\ \ for.a=b=c$$

1384. *If* $x,y,z>0$, $x^2+y^2+z^2=3$ *then*

$$\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}-\frac{1}{x+y+z}\geq\frac{17}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{3}\overset{AM-HM}{\geq}\frac{3}{x+y+z}\ or$$
$$\frac{1}{x+y+z}\leq\frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ (1)$$

$$\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\geq\frac{3(x^2+y^2+z^2)}{x+y+z}\ (2)$$

$Proof$:

$$(x+y+z)\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)\geq 3(x^2+y^2+z^2)or$$
$$\sum\frac{x^3}{y}+\sum\frac{xy^2}{z}\geq 2\sum x^2\ \ or$$

$$\sum\frac{x^4}{yx}+\sum\frac{x^2y^2}{zx}\geq 2\sum x^2\ \ or$$

$$\frac{(\sum x^2)^2}{\sum xy}+\frac{(\sum xy)^2}{\sum xy}\geq 2\sum x^2\ \ (Bergstrom)$$

$$or\ \left(\sum x^2\right)^2+\left(\sum xy\right)^2-2\left(\sum xy\right)\left(\sum x^2\right)\geq 0\ or$$

$$\left(\sum x^2-\sum xy\right)^2\geq 0\ \ (Proof\ complete)$$

$$(x+y+z)^2\overset{CBS}{\leq}3(x^2+y^2+z^2)\ or$$
$$(x+y+z)\leq\sqrt{3(x^2+y^2+z^2)}\ (3)$$

$$\frac{x^2+1}{y}+\frac{y^2+1}{z}+\frac{z^2+1}{x}-\frac{1}{x+y+z}=$$

$$=\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)+\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right)-\left(\frac{1}{x+y+z}\right)\geq$$

$$\overset{(1)\&(2)}{\geq}\frac{3(x^2+y^2+z^2)}{x+y+z}+\left(\frac{1}{y}+\frac{1}{z}+\frac{1}{x}\right)-\frac{1}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

$$=\frac{3(x^2+y^2+z^2)}{x+y+z}+\frac{8}{9}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq$$

$$\overset{Bergstrom}{\geq}\frac{3(x^2+y^2+z^2)}{x+y+z}+\frac{8}{9}\frac{(1+1+1)^2}{x+y+z}\overset{(3)}{\geq}$$

$$\geq\frac{3(x^2+y^2+z^2)}{\sqrt{3(x^2+y^2+z^2)}}+\frac{8}{\sqrt{3(x^2+y^2+z^2)}}=$$

$$=\frac{3.3}{\sqrt{3.3}}+\frac{8}{\sqrt{3.3}}(since\ x^2+y^2+z^2=3)=3+\frac{8}{3}=\frac{17}{3}$$

$$Equality\ holds\ for\ x=y=z=1$$

1385. If $a,b,c>0$ and $abc=1$, then prove that :

$$a^3+b^3+c^3+6\geq(a+b+c)^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z\Rightarrow x+y-z=2c>0$,
$y+z-x=2a>0$ and $z+x-y=2b>0\Rightarrow x+y>z, y+z>x, z+x>y$
$\Rightarrow x,y,z$ form sidesof a triangle with semiperimeter, circumradius and inradius
$=s,R,r$ (say); so $2\sum_{cyc}a=\sum_{cyc}x=2s\Rightarrow\sum_{cyc}a=s\rightarrow(1)$

$$\Rightarrow a=s-x, b=s-y, c=s-z\therefore abc=r^2s\rightarrow(2)$$

$$\text{Such substitutions}\Rightarrow\sum_{cyc}a^3=\left(\sum_{cyc}a\right)^3-3(a+b)(b+c)(c+a)\overset{via\,(1)}{=}s^3-3xyz$$

$$=s^3-12Rrs\therefore\sum_{cyc}a^3=s^3-12Rrs\rightarrow(3)$$

$$\text{Now, }a^3+b^3+c^3+6\geq2(a+b+c)^2\overset{abc=1}{\Leftrightarrow}\sum_{cyc}a^3+6abc\geq\sqrt[3]{abc}\left(\sum_{cyc}a\right)^2$$

$$\overset{\text{via (1),(2) and (3)}}{\Leftrightarrow} (s^3-12Rrs+6r^2s)^3 \geq r^2s^7 \Leftrightarrow (s^2-12Rr+6r^2)^3 \overset{(*)}{\geq} r^2s^4 \text{ and}$$

$$\because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \geq (s^2-16Rr+5r^2)^3 \Leftrightarrow (12R+2r)s^4 - r(336R^2-48Rr-33r^2)s^2$$

$$+r^2(2368R^3-1248R^2r-96Rr^2+91r^3) \overset{(**)}{\geq} 0 \text{ and}$$

$$\because (12R+2r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \geq (12R+2r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (48R^2-8Rr+13r^2)s^2 \overset{(***)}{\geq} r(704R^3-160R^2r+76Rr^2-41r^3)$$

$$\text{Finally, } (48R^2-8Rr+13r^2)s^2 \overset{\text{Gerretsen}}{\geq} (48R^2-8Rr+13r^2)(16Rr-5r^2) \overset{?}{\geq}$$

$$r(704R^3-160R^2r+76Rr^2-41r^3) \Leftrightarrow 16t^3-52t^2+43t-6 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)(6t^2+10t(t-2)+3) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore a^3+b^3+c^3+6 \geq (a+b+c)^2 \; \forall a,b,c>0 \,|\, abc=1,$$

$$''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1386. $\forall a,b,c>0 \wedge n \in \mathbb{N}$, **prove that :**

$$\sum_{cyc} \frac{c.\sqrt{b^n+c^n}+b.\sqrt{c^n+a^n}}{b(a+b)} \geq 3\sqrt{2}(abc)^{\frac{n-2}{6}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A,B,C>0, (A+B),(B+C),(C+A) \text{ form sides of } a \text{ triangle}$$

$$(\because (A+B)+(B+C)>(C+A) \text{ and analogs}) \Rightarrow \sqrt{A+B},\sqrt{B+C},\sqrt{C+A} \text{ form}$$

$$\text{sides of } a \text{ triangle with area F (say) and } 16F^2 =$$

$$2\sum_{cyc}(A+B)(B+C)-\sum_{cyc}(A+B)^2 = 2\sum_{cyc}\left(\sum_{cyc}AB+B^2\right)-2\sum_{cyc}A^2-2\sum_{cyc}AB$$

$$=6\sum_{cyc}AB+2\sum_{cyc}A^2-2\sum_{cyc}A^2-2\sum_{cyc}AB \Rightarrow 4F=2\sqrt{\sum_{cyc}AB} \to (1)$$

$$\text{Now, } \forall x,y,z>0, \sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}} \overset{?}{\geq} \frac{\sqrt{3}}{2} \Leftrightarrow \sum_{cyc}\frac{x^2y^2}{xy(y+z)(z+x)} \underset{(*)}{\overset{?}{\geq}} \frac{3}{4}$$

$$\text{Via Bergstrom, LHS of } (*) \geq \frac{\left(\sum_{cyc}xy\right)^2}{\sum_{cyc}\left(xy\left(\sum_{cyc}xy+z^2\right)\right)} = \frac{\left(\sum_{cyc}xy\right)^2}{\left(\sum_{cyc}xy\right)^2+xyz\sum_{cyc}x}$$

$$\overset{?}{\geq}\frac{3}{4}\Leftrightarrow\left(\sum_{cyc}xy\right)^2\overset{?}{\geq}3xyz\sum_{cyc}x\rightarrow true\therefore\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}\overset{?}{\geq}\frac{\sqrt{3}}{2}\rightarrow(2)$$

$$\text{We have}:\frac{c.\sqrt{b^n+c^n}+b.\sqrt{c^n+a^n}}{b(a+b)}+\frac{c.\sqrt{a^n+b^n}+a.\sqrt{c^n+a^n}}{c(b+c)}$$
$$+\frac{b.\sqrt{a^n+b^n}+a.\sqrt{b^n+c^n}}{a(c+a)}$$
$$=\frac{\frac{\sqrt{b^n+c^n}}{b}+\frac{\sqrt{c^n+a^n}}{c}}{\frac{a+b}{c}}+\frac{\frac{\sqrt{a^n+b^n}}{a}+\frac{\sqrt{c^n+a^n}}{c}}{\frac{b+c}{a}}+\frac{\frac{\sqrt{a^n+b^n}}{a}+\frac{\sqrt{b^n+c^n}}{b}}{\frac{c+a}{b}}$$
$$=\frac{c}{a+b}.\left(\frac{\sqrt{b^n+c^n}}{b}+\frac{\sqrt{c^n+a^n}}{c}\right)+\frac{a}{b+c}.\left(\frac{\sqrt{c^n+a^n}}{c}+\frac{\sqrt{a^n+b^n}}{a}\right)$$
$$+\frac{b}{c+a}.\left(\frac{\sqrt{a^n+b^n}}{a}+\frac{\sqrt{b^n+c^n}}{b}\right)=\frac{x}{y+z}(B+C)+\frac{y}{z+x}(C+A)+\frac{z}{x+y}(A+B)$$
$$\left(\begin{matrix}x=c,y=a,z=b,\\A=\frac{\sqrt{a^n+b^n}}{a},B=\frac{\sqrt{b^n+c^n}}{b},C=\frac{\sqrt{c^n+a^n}}{c}\end{matrix}\right)$$
$$=\frac{x}{y+z}.\sqrt{B+C}^2+\frac{y}{z+x}.\sqrt{C+A}^2+\frac{z}{x+y}\overset{\text{Oppenheim}}{\geq}4F.\sqrt{\sum_{cyc}\frac{xy}{(y+z)(z+x)}}$$
$$\overset{\text{via (1) and (2)}}{\geq}2\sqrt{\sum_{cyc}AB}.\frac{\sqrt{3}}{2}=\sqrt{3\sum_{cyc}\left(\frac{\sqrt{a^n+b^n}}{a}.\frac{\sqrt{b^n+c^n}}{b}\right)}\overset{A-G}{\geq}$$
$$3.\sqrt{\sqrt[3]{\frac{(a^n+b^n)(b^n+c^n)(c^n+a^n)}{a^2b^2c^2}}}\overset{Cesaro}{\geq}3.\sqrt[6]{\frac{8(abc)^n}{(abc)^2}}=3\sqrt{2}(abc)^{\frac{n-2}{6}}$$
$$\therefore\frac{c.\sqrt{b^n+c^n}+b.\sqrt{c^n+a^n}}{b(a+b)}+\frac{c.\sqrt{a^n+b^n}+a.\sqrt{c^n+a^n}}{c(b+c)}$$
$$+\frac{b.\sqrt{a^n+b^n}+a.\sqrt{b^n+c^n}}{a(c+a)}\geq3\sqrt{2}(abc)^{\frac{n-2}{6}}\ \forall\ a,b,c>0\ \text{and}\ n\in\mathbb{N},$$
$$''='' \text{ iff } a=b=c\ (QED)$$

1387. If $x>y>z>0$, then prove that :

$$\frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z\left(\sqrt{xz}-z\right)}\geq5$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z(\sqrt{xz}-z)} \overset{\text{Bergstrom}}{\geq} \frac{(\sqrt{x}+\sqrt{z})^2}{x-y+y-z}+\frac{x^2}{8z(\sqrt{xz}-z)}$$

$$=\frac{x+z+2\sqrt{xz}}{x-z}+\frac{x^2}{8z(\sqrt{xz}-z)} \therefore \frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z(\sqrt{xz}-z)}$$

$$\geq \frac{t+1+2\sqrt{t}}{t-1}+\frac{t^2}{8(\sqrt{t}-1)} \left(t=\frac{x}{z}\right) \to (1)$$

$$\text{Let } f(t)=\frac{t+1+2\sqrt{t}}{t-1}+\frac{t^2}{8(\sqrt{t}-1)} \ \forall\, t\in(1,\infty) \text{ and then :}$$

$$f'(t)=\frac{3t^2-4t.\sqrt{t}-16}{16\sqrt{t}(\sqrt{t}-1)^2}=\frac{3m^4-4m^3-16}{16m(m-1)^2} \ (m=\sqrt{t})$$

$$=\frac{(m-2)(3m^3+2m^2+4m+8)}{16m(m-1)^2} \therefore f'(t)=0 \text{ has a unique root, it being } t=4$$

$$(\because 3m^3+2m^2+4m+8>0) \text{ and } f''(t)|_{t=4}$$

$$=\frac{t.\sqrt{t}(3t^2+8t+48)-t(9t^2+16)}{32t^2.\sqrt{t}(\sqrt{t}-1)^3}\Bigg|_{t=4}=\frac{3}{8}>0$$

$$\therefore f(t) \text{ attains a minima at } t=4 \text{ and } \because f'(t)\leq 0 \ \forall\, t\in(1,4] \text{ and}$$

$f'(t)\geq 0 \ \forall\, t\in[4,\infty) \therefore f(t)$ is $\downarrow$ on $(1,4]$ and $f(t)$ is $\uparrow$ on $[4,\infty) \therefore f(t)\geq f(4)=5$

$$\therefore \frac{t+1+2\sqrt{t}}{t-1}+\frac{t^2}{8(\sqrt{t}-1)}\geq 5 \to (2) \therefore (1) \text{ and } (2) \Rightarrow$$

$$\frac{x}{x-y}+\frac{z}{y-z}+\frac{x^2}{8z(\sqrt{xz}-z)}\geq 5 \ \forall\, x>y>z>0, ''='' \text{ iff } x=4z \text{ (QED)}$$

1388. If $x,y,z>0$ and $x\geq z$, then prove that :

$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\sqrt{\frac{z}{z+x}}\leq\sqrt{5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\sqrt{\frac{z}{z+x}}$$

$$=\sqrt{2}.\frac{x}{\sqrt{2(x^2+y^2)}}+\sqrt{2}.\frac{y}{\sqrt{2(y^2+z^2)}}+\sqrt{\frac{z}{z+x}} \overset{\text{CBS}}{\leq}$$

$$\sqrt{2+2+1}\cdot\sqrt{\frac{x^2}{2(x^2+y^2)}+\frac{y^2}{2(y^2+z^2)}+\frac{z}{z+x}}\overset{?}{\leq}\sqrt{5}$$

$$\Leftrightarrow \frac{x^2}{x^2+y^2}+\frac{y^2}{y^2+z^2}+\frac{2z}{z+x}\overset{?}{\leq}2 \Leftrightarrow \frac{2x}{z+x}-1\overset{?}{\geq}\frac{x^2y^2+z^2x^2+x^2y^2+y^4}{x^2y^2+z^2x^2+y^4+y^2z^2}-1$$

$$\Leftrightarrow \frac{x-z}{z+x}\overset{?}{\geq}\frac{y^2(x-z)(z+x)}{x^2y^2+z^2x^2+y^4+y^2z^2}$$

$$\Leftrightarrow (x-z)\left(\frac{1}{z+x}-\frac{y^2(z+x)}{x^2y^2+z^2x^2+y^4+y^2z^2}\right)\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{(x-z)(x^2y^2+z^2x^2+y^4+y^2z^2-y^2z^2-x^2y^2-2y^2zx)}{(z+x)(x^2y^2+z^2x^2+y^4+y^2z^2)}\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{(x-z)(z^2x^2+y^4-2y^2zx)}{(z+x)(x^2y^2+z^2x^2+y^4+y^2z^2)}\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{(x-z)(zx-y^2)^2}{(z+x)(x^2y^2+z^2x^2+y^4+y^2z^2)}\overset{?}{\geq}0 \to \text{true} \because x,y,z>0\wedge x\geq z$$

$$\therefore \frac{x}{\sqrt{x^2+y^2}}+\frac{y}{\sqrt{y^2+z^2}}+\sqrt{\frac{z}{z+x}}\leq\sqrt{5}\ \forall\, x,y,z>0\,|\,x\geq z\ (\text{QED})$$

1389. If $x,y,z>0$ and $x^2+y^2+z^2=1$, then prove that :

$$\frac{1}{\sqrt{x^2+xy}}+\frac{1}{\sqrt{y^2+xy}}+\frac{2\sqrt{3}}{1+z}\geq\frac{8\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{\sqrt{x^2+xy}}+\frac{1}{\sqrt{y^2+xy}}\overset{\text{Radon}}{\geq}\frac{2^{\frac{3}{2}}}{\sqrt{x^2+xy+y^2+xy}}=\frac{\sqrt{8}}{x+y}$$

$$\overset{\text{Radon}}{\geq}\frac{\sqrt{8}}{\sqrt{2(x^2+y^2)}}\overset{x^2+y^2+z^2=1}{=}\frac{2}{\sqrt{1-z^2}}$$

$$\Rightarrow \frac{1}{\sqrt{x^2+xy}}+\frac{1}{\sqrt{y^2+xy}}+\frac{2\sqrt{3}}{1+z}\overset{(*)}{\geq}\frac{2}{\sqrt{1-z^2}}+\frac{2\sqrt{3}}{1+z}$$

$$\text{Let } f(z)=\frac{1}{\sqrt{1-z^2}}+\frac{\sqrt{3}}{1+z}\ \forall z\in(0,1)\ \text{and then}:$$

$$f'(z)=\frac{z}{(1-z^2)^{\frac{3}{2}}}-\frac{\sqrt{3}}{(z+1)^2}\to(1)$$

$$\text{Now, when}: \frac{z}{(1-z^2)^{\frac{3}{2}}}-\frac{\sqrt{3}}{(z+1)^2}=0,\text{then}: \frac{z^2}{(1-z)^3(1+z)^3}=\frac{3}{(z+1)^4}$$

$$\Rightarrow 4z^3-8z^2+9z-3=0\Rightarrow(2z-1)(2z^2-3z+3)=0\ \text{and}$$

$$\because \text{discriminant of } (2z^2-3z+3)=9-24<0\ \therefore 2z^2-3z+3>0\ \text{and so,}$$

$$\frac{z}{(1-z^2)^{\frac{3}{2}}}-\frac{\sqrt{3}}{(z+1)^2}=0\Rightarrow z=\frac{1}{2}\therefore f'\left(\frac{1}{2}\right)=0\ \text{and}$$

$$f''\left(\frac{1}{2}\right)=\frac{1+3z^2}{(1-z^2)^{\frac{3}{2}}}+\frac{2\sqrt{3}}{(z+1)^3}\Bigg|_{z=\frac{1}{2}}>0\therefore f(z)\ \textit{attains a minima at}\ z=\frac{1}{2}$$

$$\text{and}\because f''(z)=\frac{1+3z^2}{(1-z^2)^{\frac{3}{2}}}+\frac{2\sqrt{3}}{(z+1)^3}>0\ \forall z\in(0,1)\therefore f'(z)\ \text{is}\uparrow\ \text{on}\ (0,1)$$

$$\therefore f'(z)=0\ \textit{has a unique root, it being}\ z=\frac{1}{2}\ \textit{and so,}\ f(z)\ \textit{never attains a maxima}$$

$$\Rightarrow f(z)\geq f\left(\frac{1}{2}\right)=\frac{1}{\sqrt{1-z^2}}+\frac{\sqrt{3}}{1+z}\Bigg|_{z=\frac{1}{2}}=\frac{4\sqrt{3}}{3}\Rightarrow\frac{2}{\sqrt{1-z^2}}+\frac{2\sqrt{3}}{1+z}\overset{(**)}{\geq}\frac{8\sqrt{3}}{3}$$

$$\therefore (*),(**)\Rightarrow\frac{1}{\sqrt{x^2+xy}}+\frac{1}{\sqrt{y^2+xy}}+\frac{2\sqrt{3}}{1+z}\geq\frac{8\sqrt{3}}{3}$$

$$\forall x,y,z>0\,\big|\,x^2+y^2+z^2=1,''='' \text{ iff } \left(x=y=\frac{\sqrt{6}}{4},z=\frac{1}{2}\right)\ \text{(QED)}$$

1390. If $a,b>0, a+b=2$ and $0\leq\lambda\leq 1$, then :

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}\geq\frac{2}{\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}\geq\frac{2}{\lambda+1}\Leftrightarrow\left(\frac{1}{a^3+\lambda}-\frac{1}{\lambda+1}\right)+\left(\frac{1}{b^3+\lambda}-\frac{1}{\lambda+1}\right)\geq 0$$

$$\Leftrightarrow \frac{1-a^3}{a^3+\lambda}+\frac{1-b^3}{b^3+\lambda}\geq 0\ \left(\because \frac{1}{\lambda+1}>0\ as\ 0\leq\lambda\leq 1\right)$$

$$\Leftrightarrow \frac{b^3-a^3b^3+\lambda-\lambda a^3+a^3-a^3b^3+\lambda-\lambda b^3}{(a^3+\lambda)(b^3+\lambda)}\geq 0$$

$$\Leftrightarrow a^3+b^3-2a^3b^3+2\lambda-\lambda(a^3+b^3)\overset{(*)}{\geq}0$$

$$\text{Now},2=a+b\overset{A-G}{\geq}2.\sqrt{ab}\Rightarrow ab\leq 1\Rightarrow a^2b^2\leq 1\Rightarrow 2a^3b^3\leq 2ab$$
$$\Rightarrow \text{LHS of }(*)\geq a^3+b^3-2ab+2\lambda-\lambda(a^3+b^3)$$
$$\overset{a+b=2}{=}2(a^2-ab+b^2)-2ab+\frac{2\lambda}{4}(a+b)^2-2\lambda(a^2-ab+b^2)$$
$$=2(a-b)^2+\frac{\lambda}{2}(a^2+2ab+b^2-4a^2+4ab-4b^2)=2(a-b)^2-\frac{3\lambda}{2}.(a-b)^2$$
$$=\frac{(a-b)^2}{2}(4-3\lambda)\geq 0\because 4-3\lambda=1+3(1-\lambda)\overset{0\leq\lambda\leq 1}{\geq}1>0$$
$$\therefore \frac{1}{a^3+\lambda}+\frac{1}{b^3+\lambda}\geq\frac{2}{\lambda+1}\ \forall a,b>0\,|\,a+b=2\wedge 0\leq\lambda\leq 1,$$
$$''='' \text{ iff } a=b=1\ (\text{QED})$$

1391. *If $a,b,c>0$, $a^2+b^2+c^2=3$, $\lambda\geq 0$ then*

$$\sum\frac{1}{a+\lambda b}\geq\frac{1}{(\lambda+1)(\sqrt{\lambda+1})}\sum\sqrt{a^2+\lambda b^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$(a+b+c)^2\overset{CBS}{\leq}3(a^2+b^2+c^2)=9\ (as\ ,a^2+b^2+c^2=3)or$$

$$(a+b+c)\leq 3\ (1)$$

$$R.H.S=\frac{1}{(\lambda+1)(\sqrt{\lambda+1})}\sum\sqrt{a^2+\lambda b^2}\overset{CBS}{\leq}$$

$$\leq\frac{1}{(\lambda+1)(\sqrt{\lambda+1})}\sqrt{3(a^2+\lambda b^2+b^2+\lambda c^2+c^2+\lambda a^2)}=$$

$$=\frac{1}{(\lambda+1)\left(\sqrt{\lambda+1}\right)}\sqrt{3(\lambda+1)(a^2+b^2+c^2)}=\frac{1}{(\lambda+1)\left(\sqrt{\lambda+1}\right)}\sqrt{3(\lambda+1)(3)}$$

$$=\frac{3}{\lambda+1}\ (since\ a^2+b^2+c^2=3)\ (2)$$

$$L.H.S=\sum\frac{1}{a+\lambda b}=\sum\frac{1^2}{a+\lambda b}\overset{Bergstrom}{\geq}\frac{(1+1+1)^2}{(a+b+c)(\lambda+1)}\geq$$

$$\overset{(1)}{\geq}\frac{9}{3(\lambda+1)}=\frac{3}{(\lambda+1)}\ (3)$$

$$from(2)\&(3)we\ can\ \ say\ L.H.S\geq R.H.S\ or$$

$$\sum\frac{1}{a+\lambda b}\geq\frac{1}{(\lambda+1)\left(\sqrt{\lambda+1}\right)}\sum\sqrt{a^2+\lambda b^2}$$

$$Equality\ for\ \ a=b=c=1$$

1392. If $a,b,c>0$ and $abc=1$, then :

$$\sum_{cyc}\frac{a^2+bc}{a+b}\geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2+bc}{a+b}=\sum_{cyc}\frac{a^2-b^2+b^2}{a+b}+\sum_{cyc}\frac{bc}{a+b}=$$

$$=\sum_{cyc}\frac{(a+b)(a-b)}{a+b}+\sum_{cyc}\frac{b^2+bc}{a+b}=$$

$$=\sum_{cyc}(a-b)+\frac{b(b+c)}{a+b}+\frac{c(c+a)}{b+c}+\frac{a(a+b)}{c+a}\overset{A-G}{\geq}$$

$$\geq 3\sqrt[3]{\frac{abc(b+c)(c+a)(a+b)}{(a+b)(b+c)(c+a)}}\overset{abc=1}{=}3\ \therefore\sum_{cyc}\frac{a^2+bc}{a+b}\geq 3$$

$$\forall\ a,b,c>0\ |\ abc=1,''='' \ \text{iff}\ a=b=c=1\ (\text{QED})$$

1393. If $a,b,c>0$, then prove that :

$$\sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3}\le\frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}=$$

$$=\frac{2a^2}{a+b}+\frac{2b^2}{b+c}+\frac{2c^2}{c+a}+2\left(\frac{(a^2-b^2)+b^2}{a+b}+\frac{(b^2-c^2)+c^2}{b+c}+\frac{(c^2-a^2)+a^2}{c+a}\right)=$$

$$=\frac{2a^2}{a+b}+\frac{2b^2}{b+c}+\frac{2c^2}{c+a}+2\sum_{cyc}(a-b)+\frac{2b^2}{a+b}+\frac{2c^2}{b+c}+\frac{2a^2}{c+a}$$

$$\therefore\frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}=\frac{2a^2+2b^2}{a+b}+\frac{2b^2+2c^2}{b+c}+\frac{2c^2+2a^2}{c+a}\to(1)$$

$$\text{Now},\frac{2b^2+2c^2}{b+c}\overset{?}{\ge}\sqrt[3]{4b^3+4c^3}\Leftrightarrow(2b^2+2c^2)^3\overset{?}{\ge}(4b^3+4c^3)(b+c)^3$$

$$\Leftrightarrow t^6-3t^5+3t^4-2t^3+3t^2-3t+1\overset{?}{\ge}0\ \left(t=\frac{b}{c}\right)\Leftrightarrow(t-1)^4(t^2+t+1)\overset{?}{\ge}0$$

$$\to\text{true}\therefore\frac{2b^2+2c^2}{b+c}\ge\sqrt[3]{4b^3+4c^3}\ \text{and analogs}\Rightarrow$$

$$\frac{2a^2+2b^2}{a+b}+\frac{2b^2+2c^2}{b+c}+\frac{2c^2+2a^2}{c+a}\ge\sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3}$$

$$\overset{\text{via (1)}}{\Rightarrow}\frac{4a^2}{a+b}+\frac{4b^2}{b+c}+\frac{4c^2}{c+a}\ge\sqrt[3]{4a^3+4b^3}+\sqrt[3]{4b^3+4c^3}+\sqrt[3]{4c^3+4a^3}$$

$$\forall a,b,c>0,''='' \text{ iff } a=b=c\ (\text{QED})$$

1394. If $a,b,c>0$, then :

$$a^4+b^4+c^4+19(a^2+b^2+c^2)>28(a+b+c)+18abc-48$$

Proposed by Khaled Abd Almouti-Syria

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^4 + 19\sum_{cyc} a^2 - 28\sum_{cyc} a + 48 - 18abc \overset{\substack{\text{Holder}\\+\\\text{A-G}}}{\geq}$$

$$\frac{1}{27}\cdot\left(\sum_{cyc} a\right)^4 + \frac{19}{3}\cdot\left(\sum_{cyc} a\right)^2 - 28\sum_{cyc} a + 48 - \frac{18}{27}\cdot\left(\sum_{cyc} a\right)^3 \overset{?}{>} 0$$

$$\Leftrightarrow t^4 - 18t^3 + 171t^2 - 756t + 1296 \overset{?}{>} 0 \left(t = \sum_{cyc} a\right)$$

$$\Leftrightarrow (t-4)^2\left(t^2 - 10t + 75\right) + 4t + 96 \overset{?}{>} 0$$

$$\Leftrightarrow (t-4)^2\left((t-5)^2 + 50\right) + 4t + 96 \overset{?}{>} 0 \to \text{true} \because t = \sum_{cyc} a > 0$$

$$\therefore a^4 + b^4 + c^4 + 19\left(a^2 + b^2 + c^2\right) > 28(a + b + c) + 18abc - 48 \text{ (QED)}$$

1395. *If $a, b, c > 0$ then:*

$$\frac{b+c}{a+\sqrt[3]{4b^3+4c^3}} + \frac{c+a}{b+\sqrt[3]{4c^3+4a^3}} + \frac{a+b}{c+\sqrt[3]{4a^3+4b^3}} \leq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\forall x, y > 0\ \frac{x^3+y^3}{2} \geq \left(\frac{x+y}{2}\right)^3 \ or, 4x^3 + 4y^3 \geq (x+y)^3 \ \ (1)$$

$$\frac{b+c}{a+\sqrt[3]{4b^3+4c^3}} + \frac{c+a}{b+\sqrt[3]{4c^3+4a^3}} + \frac{a+b}{c+\sqrt[3]{4a^3+4b^3}} =$$

$$= \sum \cdot \frac{b+c}{a+\sqrt[3]{4b^3+4c^3}} \overset{(1)}{\leq} \sum \frac{b+c}{a+\sqrt[3]{(b+c)^3}} =$$

$$= \sum \frac{b+c}{a+b+c} = \frac{2(a+b+c)}{a+b+c} = 2$$

$$Equality\ for\ a = b = c$$

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1396. If $a,b,c>0,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ then:

$$2^a+2^b+2^c\geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$2^a+2^b+2^c\overset{AM-GM}{\geq}3\cdot\sqrt[3]{2^a\cdot 2^b\cdot 2^c}=3\sqrt[3]{2^{a+b+c}}\geq$$

$$\overset{AM-HM}{\geq}3\cdot\sqrt[3]{\frac{9}{2^{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}}}=3\cdot\sqrt[3]{2^{\frac{9}{3}}}=3\cdot\sqrt[3]{2^3}=6$$

Equality holds for: $a=b=c=1$.

1397. *If* $a,b,c>0,\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2$ *then*:

$$\sum\frac{1}{\sqrt{7a^2+ab+b^2}}\leq\frac{2}{3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$7a^2+ab+b^2=6a^2+(a^2+b^2)+ab\overset{AM-GM}{\geq}$$
$$\geq 6a^2+2ab+ab=6a^2+3ab=3a(2a+b)\ \ (1)$$

$$\sum\frac{1}{\sqrt{7a^2+ab+b^2}}\overset{(1)}{\leq}\sum\frac{1}{\sqrt{3a(2a+b)}}=$$

$$=\sum\sqrt{\frac{1}{3a}\cdot\frac{1}{2a+b}}\overset{AM-GM}{\leq}\sum\frac{1}{2}\left(\frac{1}{3a}+\frac{1}{2a+b}\right)=\frac{1}{6}\sum\frac{1}{a}+\frac{1}{2}\sum\frac{1}{2a+b}=$$

$$=\frac{1}{6}\sum\frac{1}{a}+\frac{1}{2}\sum\frac{1}{a+a+b}\overset{AM-HM}{\leq}\frac{1}{6}\sum\frac{1}{a}+\frac{1}{2}\cdot\frac{1}{9}\sum\left(\frac{1}{a}+\frac{1}{a}+\frac{1}{b}\right)=$$

$$=\frac{1}{6}\cdot 2+\frac{1}{18}(2+2+2)\left(as\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=2\right)=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

$$Equality\ holds\ for\ a=b=c=\frac{3}{2}$$

1398. *If* $a,b,c>0$ *and* $abc=1$ *then*:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a^5+b^5}{c^3(a^3+b^3)}+\frac{b^5+c^5}{a^3(b^3+c^3)}+\frac{c^5+a^5}{b^3(c^3+a^3)}\geq 6$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Tapas Das-India

$$\frac{a^5+b^5}{2}\geq\frac{a^3+b^3}{2}\cdot\frac{a^2+b^2}{2}\ or$$
$$(a^5+b^5)\geq\frac{(a^3+b^3)(a^2+b^2)}{2}\ (1)$$

$$\frac{a^5+b^5}{c^3(a^3+b^3)}\overset{(1)}{\geq}\frac{(a^3+b^3)(a^2+b^2)}{2c^3(a^3+b^3)}=\frac{a^2+b^2}{2c^3}\overset{AM-GM}{\geq}\frac{2ab}{2c^3}=\frac{ab}{c^3}$$

$$Similarly,\frac{b^5+c^5}{a^3(b^3+c^3)}\geq\frac{bc}{a^3}\ and\ \frac{c^5+a^5}{b^3(c^3+a^3)}\geq\frac{ca}{b^3}$$

$$\cdot\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a^5+b^5}{c^3(a^3+b^3)}+\frac{b^5+c^5}{a^3(b^3+c^3)}+\frac{c^5+a^5}{b^3(c^3+a^3)}\geq$$

$$\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{ab}{c^3}+\frac{bc}{a^3}+\frac{ca}{b^3}\overset{AM-GM}{\geq}6\sqrt[6]{\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\cdot\frac{ab}{c^3}\cdot\frac{bc}{a^3}\cdot\frac{ca}{b^3}}=$$

$$=6\sqrt[6]{\frac{1}{a^2b^2c^2}}=6(as\ \ abc=1)$$

$$Equality\ holds\ for\ a=b=c=1$$

1399. *If* $a,b,c>0$ *and* $a^3+b^3+c^3=3$ *then*:

$$\frac{a^3}{b^2-2b+3}\cdot+\frac{2b^3}{c^3+a^2-2a-3c+7}+\frac{3c^3}{a^4+b^4+a^2-2b^2-6a+11}\leq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$b^2-2b+3=(b^2+1)-2b+2\overset{AM-GM}{\geq}2b-2b+2=2\ (1)$$

$$c^3+a^2-2a-3c+7=(c^3+1+1)+(a^2+1)-2a-3c+4 \overset{AM-GM}{\geq}$$

$$\geq 3c+2a-2a-3c+4=4 \ (2)$$

$$a^4+b^4+a^2-2b^2-6a+11=$$
$$=(a^4+a^2+1+1+1+1)+(b^4+1)-2b^2-6a+6 \overset{AM-GM}{\geq}$$

$$\geq 6a+2b^2-2b^2-6a+6=6 \ (3)$$

$$\frac{a^3}{b^2-2b+3}.+\frac{2b^3}{c^3+a^2-2a-3c+7}+\frac{3c^3}{a^4+b^4+a^2-2b^2-6a+11} \overset{(1),(2),(3)}{\leq}$$

$$\leq \frac{a^3}{2}+\frac{2b^3}{4}+\frac{3c^3}{6}=\frac{1}{2}(a^3+b^3+c^3)=\frac{3}{2}$$

$$Equality\ for\ a=b=c=1$$

1400. If $a, b>0$ and $a+b=ab$, then prove that :

$$a^b+b^a>6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a+b=ab \Rightarrow a=b(a-1)>0 \Rightarrow a>1 \text{ and similarly}, b=a(b-1)>0$$

$$\Rightarrow b>1 \therefore a,b>1 \therefore a^b+b^a \overset{\text{Bernoulli}}{>} \big(1+b(a-1)\big)+\big(1+a(b-1)\big)$$

$$=2+2ab-(a+b) \overset{a+b=ab}{=}=2+2ab-ab \therefore a^b+b^a>2+ab \rightarrow (1)$$

$$\text{Now}, ab=a+b \overset{A-G}{\geq} 2\sqrt{ab} \Rightarrow \sqrt{ab}\geq 2 \Rightarrow ab\geq 4 \rightarrow (2)$$

$$\therefore (1),(2) \Rightarrow a^b+b^a>6 \ (\text{QED})$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru