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An Application of Lifting the Exponent Theorems

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Abstract

In this article, we apply Lifting The Exponent theorems to find all ordered solutions (a, b, n) satisfying the conditions that a and b are integers, n is a positive integer, and $b^n|(a+b)^n - a^n$.

Key words and Phrases: Divisibility, difference of two powered integers, lifting the exponent theorems.

1 Introduction

Divisibility is one of most important topic in number theory. An interesting thing I encounter related to divisibility is when I consider the integer divisors of the difference of n^{th} power of two integers, let us say $(a+b)^n$ and a^n where $a, b \in \mathbb{Z}$. What will happen if b^n is a divisor of $(a+b)^n - a^n$? For which $(a, b, n) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ so that b^n becomes a divisor of $(a+b)^n - a^n$? Actually, this question arises as a generalization of one problem in Romaniuc's article [1]. The original problem is to find all $n \in \mathbb{Z}^+$ so that 2^n divides $3^n - 1$.

If we talk about the difference of n^{th} power of two integers, I remember about the important theorems which is called **Lifting The Exponent** or **LTE theorems**. LTE theorems are used to find the greatest power of a prime which can divide the difference of the same power of two integers. We usually use the sign $v_n(x)$ to denote the exponent of the highest power of $n \in \mathbb{Z}^+$ that divides $x \in \mathbb{Z}$. It is obvious that $v_n(x)$ must be a nonnegative integer.

Definition 1.1. *For every $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, $v_n(x)$ denotes the exponent of the highest power of n that divides x. It is equivalent to say that $v_n(x)$ is the greatest nonnegative integer so that $n^{v_n(x)}|x$, but $n^{v_n(x)+1} \nmid x$.*

Example 1.2. *Observe that $v_3(-18) = 2$ because $3^2| - 18$ but $3^3 \nmid -18$; $v_{10}(2024) = 0$ because $10^0|2024$ but $10^1 \nmid 2024$; and $v_8(4096) = 4$ because $8^4|4096$ but $8^5 \nmid 4096$.*

Example 1.3. *For all $n \in \mathbb{Z}^+$, we have $v_n(0) = \infty$. It is because $n^k|0$, $\forall k \in \mathbb{N}_0$.*

LTE theorems are used on the case of $v_n()$ when $n = p$ is a prime. Applying LTE theorems is a useful method to find the general solutions $(a, b, n) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ so that $(a + b)^n - a^n$ is divisible by b^n with the assumption that b has a prime divisor. Some examples of LTE theorems we use here are
(i). Let x, y be odd integers and n be an odd positive integer, so we have

$$v_2(x^n - y^n) = v_2(x - y).$$

(ii). Let x, y be odd integers and n be an even positive integer, so we have

$$v_2(x^n - y^n) = v_2(x + y) + v_2(x - y) + v_2(n) - 1.$$

(iii). Let p be an odd prime. Let x, y be integers with $gcd(p, x) = gcd(p, y) = 1$ and $p|x - y$. Then, for all $n \in \mathbb{Z}^+$,

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n).$$

For their proofs and many other theorems, these are written in Section 2.

After so many complicated computations, finally we can list all the solutions $(a, b, n) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ so that $(a + b)^n - a^n$ is divisible by a^n. List of all these solutions can be seen in Section 3.

2 Some Theorems About Lifting The Exponent

Theorem 2.1. *Let x and y be odd integers. So, for all odd positive integers n, we have*

$$v_2(x^n - y^n) = v_2(x - y).$$

Proof. If $x = y$, then $v_2(x^n - y^n) = v_2(x - y) = \infty$.
The next one is for $x \neq y$. Observe that $x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1}$ and $\sum_{k=0}^{n-1} x^k y^{n-k-1} \equiv \sum_{k=0}^{n-1} 1 = n \equiv 1 \pmod 2$. Therefore,

$$v_2(x^n - y^n) = v_2(x - y) + v_2\left(\sum_{k=0}^{n-1} x^k y^{n-k-1}\right) = v_2(x - y).$$

□

Theorem 2.2. *Let x and y be odd integers and n be a positive integer. Then,*

$$v_2(x^{2^n} - y^{2^n}) = v_2(x^2 - y^2) + n - 1.$$

Proof. When $x = y$, it is clear that $v_2(x^{2^n} - y^{2^n}) = v_2(x^2 - y^2) + n - 1 = \infty$.
The next case is for $x \neq y$. Suppose that $S(n)$ is the statement that $v_2(x^{2^n} - y^{2^n}) = v_2(x^2 - y^2) + n - 1$. We will show that $S(n)$ is true for all $n \in \mathbb{Z}^+$. It is trivial that $S(1)$ is true. Assume that $S(k)$ is true for some $k \in \mathbb{Z}^+$, then we have $v_2(x^{2^k} - y^{2^k}) = v_2(x^2 - y^2) + k - 1$. Since x and y are odd, then $x^{2^k} \equiv y^{2^k} \equiv 1 \pmod 4$. It implies $x^{2^k} + y^{2^k} \equiv 2 \pmod 4$ and therefore $v_2(x^{2^k} + y^{2^k}) = 1$. Consequently,

$$\begin{aligned} v_2(x^{2^{k+1}} - y^{2^{k+1}}) = v_2((x^{2^k} + y^{2^k})(x^{2^k} - y^{2^k})) &= v_2(x^{2^k} + y^{2^k}) + v_2(x^{2^k} - y^{2^k}) \\ &= 1 + v_2(x^2 - y^2) + k - 1 \\ &= v_2(x^2 - y^2) + k \end{aligned}$$

and $S(k+1)$ is also true. By induction, we get that $S(n)$ is true for all $n \in \mathbb{Z}^+$ and hence the theorem is proven. □

Theorem 2.3. *Let x and y be odd integers. For all even positive integers n, we have*

$$v_2(x^n - y^n) = v_2(x + y) + v_2(x - y) + v_2(n) - 1.$$

Proof. When $x = y$: we have $v_2(x^n - y^n) = v_2(x^2 - y^2) + n - 1 = \infty$.
When $x \neq y$: Suppose that $n = n_1 \cdot 2^{n_2}$ where $n_1, n_2 \in \mathbb{Z}^+$, and n_1 is odd. By Theorem 2.1 and 2.2,

$$\begin{aligned} v_2(x^n - y^n) = v_2((x^{2^{n_2}})^{n_1} - (y^{2^{n_2}})^{n_1}) = v_2(x^{2^{n_2}} - y^{2^{n_2}}) &= v_2(x^2 - y^2) + n_2 - 1 \\ &= v_2((x + y)(x - y)) + v_2(n_1 \cdot 2^{n_2}) - 1 \\ &= v_2(x + y) + v_2(x - y) + v_2(n) - 1. \end{aligned}$$

□

Theorem 2.4. *Let p be an odd prime and x, y be integers with $gcd(p,x) = gcd(p,y) = 1$ and $p|x-y$. For every $n \in \mathbb{Z}^+$ with $gcd(n,p) = 1$, we have*

$$v_p(x^n - y^n) = v_p(x-y).$$

Proof. If $x = y$, then $x_p(x^n - y^n) = v_2(x-y) = \infty$.
If $x \neq y$: Consider

$$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-k-1}.$$

Since $p|x-y$ then $y \equiv x \pmod{p}$. Consequently,

$$\sum_{k=0}^{n-1} x^k y^{n-k-1} \equiv \sum_{k=0}^{n-1} x^k x^{n-k-1} \equiv \sum_{k=0}^{n-1} x^{n-1} \equiv nx^{n-1} \pmod{p}.$$

The fact $gcd(n,p) = gcd(x,p) = 1$ implies that $gcd(nx^{n-1}, p) = 1$, so $v_p\left(\sum_{k=0}^{n-1} x^k y^{n-k-1}\right) = 0$. Finally,

$$v_p(x^n - y^n) = v_p(x-y) + v_p\left(\sum_{k=0}^{n-1} x^k y^{n-k-1}\right) = v_p(x-y).$$

□

Theorem 2.5. *If p is odd prime and $n \in \mathbb{Z}^+$, then*

$$p^{n+1} \Big| \binom{p^n}{k} p^{k-1}, \ \forall k \in \{2, 3, 4, \ldots, p^n\}.$$

Proof. We have the fact that $\binom{p^n}{k+1}p^k = \binom{p^n}{k}p^{k-1} \cdot \frac{(p^n-k)p}{k+1}$ for every $k = 1, 2, 3, \ldots, p^n - 1$. So we have, for every $k = 1, 2, 3, \ldots, p^n - 1$,

$$\begin{aligned} v_p\left(\binom{p^n}{k+1}p^k\right) &= v_p\left(\binom{p^n}{k}p^{k-1}\right) + v_p(p^n - k) + v_p(p) - v_p(k+1) \\ &= v_p\left(\binom{p^n}{k}p^{k-1}\right) + v_p(k) + 1 - v_p(k+1) \end{aligned}$$

$$\Longrightarrow \quad v_p\left(\binom{p^n}{k+1}p^k\right) - v_p\left(\binom{p^n}{k}p^{k-1}\right) = 1 + v_p(k) - v_p(k+1).$$

Let $f(k) = v_p\left(\binom{p^n}{k}p^{k-1}\right)$ for every $k = 1, 2, 3, \ldots, p^n - 1$. So we have

$$f(k+1) - f(k) = 1 + v_p(k) - v_p(k+1), \quad \forall k = 1, 2, 3, \ldots, p^n - 1.$$

Consequently, for all $k = 2, 3, \ldots, p^n$,

$$f(k) - f(1) \;=\; \sum_{i=1}^{k-1} f(i+1) - f(i) \;=\; \sum_{i=1}^{k-1} 1 + v_p(i) - v_p(i+1) \;=\; k - 1 + v_p(1) - v_p(k)$$

$$\begin{aligned} \Longrightarrow \qquad f(k) \;&=\; f(1) + k - 1 - v_p(k) \;=\; v_p\left(\binom{p^n}{1}\right) + k - 1 - v_p(k) \\ &=\; n + k - 1 - v_p(k). \end{aligned}$$

Over $k \in \{2, 3, 4 \ldots, p^n\}$, if $v_p(k) = 0$ then $k - v_p(k) = k \geq 2$; and if $v_p(k) \geq 1$ then $k - v_p(k) \geq p^{v_p(k)} - v_p(k) \geq p - 1 \geq 2$. We obtain

$$f(k) \;=\; n + k - 1 - v_p(k) \geq n + 1, \quad \forall k = 2, 3, 4, \ldots, p^n$$

and it implies that p^{n+1} divides $\binom{p^n}{k} p^{k-1}$ for all $k = 2, 3, \ldots, p^n$. □

Theorem 2.6. *Let p be an odd prime. Let x and y be integers with $gcd(p, x) = gcd(p, y) = 1$ and $p|x - y$. For all $n \in \mathbb{Z}^+$, we have*

$$v_p(x^{p^n} - y^{p^n}) = v_p(x - y) + n.$$

Proof. If $x = y$, then $v_p(x^{p^n} - y^{p^n}) = v_p(x - y) + n = \infty$.
If $x \neq y$, suppose that $x - y = t \cdot p^m$ for some $m \in \mathbb{Z}^+$ and $t \in \mathbb{Z}$ with $gcd(t, p) = 1$. By using binomial theorem, $\forall \alpha \in \mathbb{Z}^+$,

$$x^\alpha = ((x - y) + y)^\alpha = \sum_{k=0}^{\alpha} \binom{\alpha}{k} (x - y)^k y^{\alpha - k} = y^\alpha + \sum_{k=1}^{\alpha} \binom{\alpha}{k} (x - y)^k y^{\alpha - k},$$

and then

$$\frac{x^\alpha - y^\alpha}{x - y} \;=\; \frac{1}{x - y} \sum_{k=1}^{\alpha} \binom{\alpha}{k} (x - y)^k y^{\alpha - k} \;=\; \sum_{k=1}^{\alpha} \binom{\alpha}{k} (x - y)^{k-1} y^{\alpha - k} \tag{1}$$

Replacing α in (1) by p^n yields that

$$x^{p^n} - y^{p^n} = (x - y) \sum_{k=1}^{p^n} \binom{p^n}{k} (x - y)^{k-1} y^{p^n - k} = (x - y) \sum_{k=1}^{p^n} \binom{p^n}{k} y^{p^n - k} t^{k-1} p^{m(k-1)}, \quad \forall n \in \mathbb{Z}^+$$

and so we have

$$v_p(x^{p^n} - y^{p^n}) = v_p(x - y) + v_p\left(\sum_{k=1}^{p^n} \binom{p^n}{k} y^{p^n - k} t^{k-1} p^{m(k-1)}\right) \tag{2}$$

Consider the expression

$$\sum_{k=1}^{p^n} \binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)} = p^n y^{p^n-1} + \sum_{k=2}^{p^n} \binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)}$$

Every $k = 2, 3, \ldots, p^n$; $\binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)}$ is divisible by $\binom{p^n}{k} p^{k-1}$ and by Theorem 2.5,

$$p^{n+1} \Big| \binom{p^n}{k} p^{k-1} \Big| \binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)}.$$

We obtain

$$p^{n+1} \Big| \sum_{k=2}^{p^n} \binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)}.$$

Since $v_p(p^n y^{p^n-1}) = n$, it implies

$$v_p\left(\sum_{k=1}^{p^n} \binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)}\right) = v_p\left(p^n y^{p^n-1} + \sum_{k=2}^{p^n} \binom{p^n}{k} y^{p^n-k} t^{k-1} p^{m(k-1)}\right) = n.$$

By equation (2), we get $v_p(x^{p^n} - y^{p^n}) = v_p(x-y) + n$. □

Theorem 2.7. *Let p be an odd prime. Let x and y be integers with $gcd(p,x) = gcd(p,y) = 1$ and $p|x-y$. For all $n \in \mathbb{Z}^+$, we have*

$$v_p(x^n - y^n) = v_p(x-y) + v_p(n).$$

Proof. Let $n = u_1 p^{u_2}$ where $u_1 \in \mathbb{Z}^+$, $u_2 \in \mathbb{N}_0$, and $gcd(p, u_1) = 1$. By Theorem 2.4 and 2.6, we get

$$\begin{aligned} v_p(x^n - y^n) = v_p(x^{u_1 p^{u_2}} - y^{u_1 p^{u_2}}) = v_p(x^{p^{u_2}} - y^{p^{u_2}}) &= v_p(x-y) + u_2 \\ &= v_p(x-y) + v_p(u_1 p^{u_2}) \\ &= v_p(x-y) + v_p(n). \end{aligned}$$

□

3 Main Problem and Its Solutions

Problem 1. *Find all ordered solutions* $(a,b,n) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^+$ *in such a way that* $(a+b)^n - a^n$ *is divisible by* b^n.

Solution. We divide this problem into 3 cases as follows.
Case 1: If $b=0$, then all $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ satisfy the condition.

Case 2: If $a=0$, then all $b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ satisfy the condition.

Case 3: If $a, b \neq 0$. Let $gcd(a,b)=d$ and $a = da_1$, $b = db_1$ where $a_1, b_1 \in \mathbb{Z}\backslash\{0\}$, $d \in \mathbb{Z}^+$, and $gcd(a_1,b_1)=1$. So we have

$$b^n | (a+b)^n - a^n \iff d^n b_1^n | d^n(a_1+b_1)^n - d^n a_1^n \iff b_1^n | (a_1+b_1)^n - a_1^n \tag{3}$$

We can see that all possible solutions of (a,b,n) here are (da_1, db_1, n) for all $d \in \mathbb{Z}^+$ and all triples $(a_1,b_1,n) \in \mathbb{Z}\backslash\{0\} \times \mathbb{Z}\backslash\{0\} \times \mathbb{Z}^+$ satisfying (3) and $gcd(a_1,b_1)=1$.
For the Case 3, we divide it into 2 following subcases:

1. If b_1 has an odd prime divisor. Let p be an odd prime divisor of b_1. Suppose that $v_p(b_1) = \alpha \in \mathbb{Z}^+$. So by (3) and LTE theorem (Theorem 2.7), we have

$$n\alpha = v_p(b_1^n) \leq v_p((a_1+b_1)^n - a_1^n) = v_p(b_1) + v_p(n) = \alpha + v_p(n)$$

$$\implies \quad v_p(n) \geq (n-1)\alpha.$$

   We obtain that $p^{(n-1)\alpha} | n$. It is clear that $n=1$ satisfies the condition $p^{(n-1)\alpha}|n$. In fact $(a_1,b_1,n) = (x,y,1), \forall x,y \in \mathbb{Z}\backslash\{0\}$ with $gcd(x,y)=1$ and y has an odd prime divisor, satisfy the condition $b^n|(a+b)^n - a^n$. If $n \geq 2$, we can check by induction that $p^{(n-1)\alpha} > n > 0$ and it is impossible to have $p^{(n-1)\alpha}|n$.

2. If b_1 does not have an odd prime divisor, so $|b_1| = 1$ or $|b_1| = 2^x$ for some $x \in \mathbb{Z}^+$.

   It is obvious that when $|b_1|=1$, every $a_1 \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ satisfy $b_1^n|(a_1+b_1)^n - a_1^n$ universally.

   When $|b_1| = 2^x$ for some $x \in \mathbb{Z}^+$, since $gcd(a_1,b_1)=1$ and b_1 is even, so a_1 is odd. Observe that

$$2^{xn} \mid b_1^n \mid (a_1+b_1)^n - a_1^n.$$

   and we have

$$xn = v_2(2^{xn}) \leq v_2((a_1+b_1)^n - a_1^n) \tag{4}$$

If n is odd, by Theorem 2.1, we have $v_2((a_1+b_1)^n - a_1^n) = v_2(b_1) = x$. By (4), it implies that $n = 1$. The solutions for this (sub)subcase are

$$(a_1, b_1, n) = (a_1', \pm 2^x, 1),\ \forall a_1' \in 2\mathbb{Z}+1,\ \forall x \in \mathbb{Z}^+.$$

If n is even; by LTE theorem (Theorem 2.3),

$$v_2((a_1+b_1)^n - a_1^n) = v_2(b_1)+v_2(2a_1+b_1)+v_2(n)-1 = x-1+v_2(2a_1+b_1)+v_2(n) \tag{5}$$

By (4) and (5), we obtain

$$v_2(n) \geq (n-1)x + 1 - v_2(2a_1 + b_1) \tag{6}$$

Notice that $2a_1 + b_1 = 2a_1 \pm 2^x$.
When $x \geq 2$, we have that $v_2(2a_1 \pm 2^x) = 1$; by (6) we get $v_2(n) \geq (n-1)x$ and $2^{(n-1)x}|n$. For $n = 2$, we have $2^x|2$, no solution. For $n \geq 4$, we have $2^{(n-1)x} \geq 2^{n-1} > n$ and the condition $2^{n-1}|n$ implies no solution.
When $x = 1$, we have $b_1 = \pm 2$ and $2a_1 + b_1 = 2a_1 \pm 2$. Let $n = 2^m t_1$ for some $m \in \mathbb{Z}^+$ and $t_1 \in 2\mathbb{Z}^+ - 1$. If $b_1 = 2$, the condition (6) implies that

$$v_2(2a_1 + 2) \geq 2^m t_1 - m.$$

For $m = t_1 = 1$, we get $n = 2$ and $v_2(2a_1 + 2) \geq 1$ which is true for any general odd a_1. This implies the solutions $(a_1, b_1, n) = (a_1', 2, 2)$ for all $a_1' \in 2\mathbb{Z}+1$.
For $(m, t_1) \neq (1, 1)$, we have $2^m t_1 - m \geq 2$. Therefore, $v_2(2a_1 + 2) \geq 2^m t_1 - m$ implies $2a_1 + 2 = 2^{2^m t_1 + t_2 - m} t_3$ for some $t_2 \in \mathbb{N}_0$ and $t_3 \in 2\mathbb{Z}+1$. Consequently, $a_1 = 2^{2^m t_1 + t_2 - m - 1} t_3 - 1$ for some $t_2 \in \mathbb{N}_0$ and $t_3 \in 2\mathbb{Z}+1$.
It is also similar when $b_1 = -2$. If $b_1 = -2$, the condition (6) implies that

$$v_2(2a_1 - 2) \geq 2^m t_1 - m.$$

For $m = t_1 = 1$, we get $n = 2$ and $v_2(2a_1 - 2) \geq 1$ which is true for any general odd a_1. This implies the solutions $(a_1, b_1, n) = (a_1', -2, 2)$ for all $a_1' \in 2\mathbb{Z}+1$.
For $(m, t_1) \neq (1, 1)$, we have $2^m t_1 - m \geq 2$. Therefore, $v_2(2a_1 - 2) \geq 2^m t_1 - m$ implies $2a_1 - 2 = 2^{2^m t_1 + t_2 - m} t_3$ for some $t_2 \in \mathbb{N}_0$ and $t_3 \in 2\mathbb{Z}+1$. Consequently, $a_1 = 2^{2^m t_1 + t_2 - m - 1} t_3 + 1$ for some $t_2 \in \mathbb{N}_0$ and $t_3 \in 2\mathbb{Z}+1$.

In conclusion, we can list that all possible ordered solutions $(a, b, n) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ satisfying $b^n|(a+b)^n - a^n$ are

(i). $(a, b, n) = (x_1, 0, x_2)$, $\forall x_1 \in \mathbb{Z}$, $\forall x_2 \in \mathbb{Z}^+$.

(ii). $(a, b, n) = (0, x_1, x_2)$, $\forall x_1 \in \mathbb{Z}$, $\forall x_2 \in \mathbb{Z}^+$.

(iii). $(a,b,n) = (da_1, db_1, 1)$, $\forall d \in \mathbb{Z}^+$, $\forall a_1, b_1 \in \mathbb{Z}\backslash\{0\}$ with $gcd(a_1, b_1) = 1$ and b_1 has an odd prime divisor.

(iv). $(a,b,n) = (da_1, \pm d, m)$, $\forall d \in \mathbb{Z}^+$, $\forall a_1 \in \mathbb{Z}\backslash\{0\}$, $\forall m \in \mathbb{Z}^+$.

(v). $(a,b,n) = (da_1, \pm 2^x d, 1)$, $\forall d \in \mathbb{Z}^+$, $\forall a_1 \in 2\mathbb{Z}+1$, $\forall x \in \mathbb{Z}^+$.

(vi). $(a,b,n) = (da_1, \pm 2d, 2)$, $\forall d \in \mathbb{Z}^+$, $\forall a_1 \in 2\mathbb{Z}+1$.

(vii). $(a,b,n) = ((2^{2^m t_1 + t_2 - m - 1} t_3 - 1)d, 2d, 2^m t_1)$, $\forall d \in \mathbb{Z}^+$, $\forall t_1 \in 2\mathbb{Z}^+ - 1$, $\forall t_2 \in \mathbb{N}_0$, $\forall t_3 \in 2\mathbb{Z}+1$, $\forall m \in \mathbb{Z}^+$ where $(m, t_1) \neq (1,1)$.

(viii). $(a,b,n) = ((2^{2^m t_1 + t_2 - m - 1} t_3 + 1)d, -2d, 2^m t_1)$, $\forall d \in \mathbb{Z}^+$, $\forall t_1 \in 2\mathbb{Z}^+ - 1$, $\forall t_2 \in \mathbb{N}_0$, $\forall t_3 \in 2\mathbb{Z}+1$, $\forall m \in \mathbb{Z}^+$ where $(m, t_1) \neq (1,1)$.

These lists can be verified again that all of them satisfies the condition $b^n|(a+b)^n - a^n$ and hence they are the overall solutions of $(a,b,n) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^+$ satisfying $b^n|(a+b)^n - a^n$.

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