On some means that refine the inequality between the arithmetic mean and the power mean

Dorin Marghidanu

In this note we will introduce and study a class of means - presented both in their form binary as well as in their n-ary form. These means will perform refinements of a well-known inequalities that occur between the arithmetic mean and quadratic or power mean - associated to two, respectively n real, positive numbers.

Keywords : *arithmetic mean, mean power, Jensen's inequality, refinement*

Mathematics Subject Classification : *26D15*

A. *The binary case*

1. *Definition*

If $a, b \in \mathbb{R}_{\geq 0}$ are two given numbers and $p_1, p_2 \in [0, 1]$ are two variable numbers such that $p_1 + p_2 = 1$, we will introduce the binary expression ,

$$\mathcal{M}_2(a, b\,; p_1, p_2) := \frac{\sqrt{p_1 a^2 + p_2 b^2} + \sqrt{p_1 b^2 + p_2 a^2}}{2} \quad . \tag{1}$$

Relative to the previously introduced expression $\mathcal{M}_2$, we have the following ,

2. *Proposition*

If $a, b, p_1, p_2 \in \mathbb{R}_{\geq 0}$, such that $p_1 + p_2 = 1$, să se demonstreze că are loc dubla inegalitate ,

$$\frac{a+b}{2} \leq \mathcal{M}_2(a, b\,; p_1, p_2) \leq \sqrt{\frac{a^2+b^2}{2}} \quad . \tag{2}$$

Proof

With the notations : $\mathbf{R}_1 = \sqrt{p_1 a^2 + p_2 b^2}$, $\mathbf{R}_2 = \sqrt{p_1 b^2 + p_2 a^2}$, we have immediately ,

$$\mathbf{R}_1^2 + \mathbf{R}_2^2 = p_1 a^2 + p_2 b^2 + p_1 b^2 + p_2 a^2 = (p_1 + p_2)(a^2 + b^2) = a^2 + b^2 \quad . \tag{3}$$

Using the *C-B-S inequality* and the condition relation , we also have ,

$$\mathbf{R}_1 \mathbf{R}_2 = \sqrt{\left(\sqrt{p_1}\, a\right)^2 + \left(\sqrt{p_2}\, b\right)^2} \cdot \sqrt{\left(\sqrt{p_1}\, b\right)^2 + \left(\sqrt{p_2}\, a\right)^2} \geq p_1 ab + p_2 ab = (p_1 + p_2)\, ab = ab \quad ,$$

(with equality if and only if $\dfrac{\sqrt{p_1}\, a}{\sqrt{p_1}\, b} = \dfrac{\sqrt{p_2}\, b}{\sqrt{p_2}\, a} \Leftrightarrow a^2 = b^2 \Leftrightarrow a = b$,

when $p_1, p_2 \neq 0$; equality is also obtained when $p_1 = 0$, $p_2 = 1$ sau $p_1 = 0$, $p_2 = 1$) . So ,

$$R_1 R_2 \geq a\, b\ , \tag{4}$$

To prove the first inequality, we note that:

$$(R_1 + R_2)^2 = R_1^2 + R_2^2 + 2\,R_1 R_2 \geq a^2 + b^2 + 2ab = (a+b)^2\ , \tag{5}$$

mean ,

$$R_1 + R_2 \geq a + b\ . \tag{6}$$

To prove the second inequality, we use the *C-B-S inequality* and relation (3) :

$$R_1 + R_2 = 1 \cdot R_1 + 1 \cdot R_2 \leq \sqrt{1^2 + 1^2} \cdot \sqrt{R_1^2 + R_2^2} = \sqrt{2(a^2 + b^2)}\ , \tag{7}$$

hence , $\frac{1}{2} \cdot (R_1 + R_2) \leq \sqrt{\frac{a^2 + b^2}{2}}$.

Equality occurs if and only if :

$$1 / R_1 = 1 / R_2 \Leftrightarrow R_1 = R_2 \Leftrightarrow p_1 a^2 + p_2 b^2 = p_1 b^2 + p_2 a^2 \Leftrightarrow$$

$$\Leftrightarrow (p_1 - p_2)(a^2 - b^2) = 0 \Leftrightarrow p_1 = p_2 (= 1/2) \text{ or } a = b\ .$$

We recall that a ***(binary) mean*** in $S \subset \mathbb{R}$ is a function $M : S \times S \longrightarrow S$, with the property of ***internality*** : $\min\{a, b\} \leq M(a, b) \leq \max\{a, b\}$.

Specific to many types of means are the properties (see e.g.: [1] , [3] , [4] , [8] , [9]) :

– *of* ***symmetry*** : if $M(a, b) = M(b, a)$, $(\forall)\, a, b \in S$;

– *of* ***homogeneity*** : if $M(k\,a, k\,b) = k\,M(b, a)$, $(\forall)\, a, b \in S$, $k \in \mathbb{R}_{>0}$.

3. *Corollary*

The expression $\mathcal{M}_2(a, b\, ; p_1, p_2) := \dfrac{\sqrt{p_1 a^2 + p_2 b^2} + \sqrt{p_1 b^2 + p_2 a^2}}{2}$ achieve through relation (2) a refinement of the inequality between the *arithmetic mean* and the *square mean* of two positive numbers a , b .

More , $\mathcal{M}_2(a, b\, ; p_1, p_2)$ is itself a (*weighted*) *mean* of the positive numbers a , b (with weights p_1 , p_2) , symmetrical and homogeneous , one .

The ***Proof*** follows from the previous Proposition . If $a \leq b$, we have obvious $\mathcal{M}_2(a, a\, ; p_1, p_2) = a$ and $\mathcal{M}_2(b, b\, ; p_1, p_2) = b$.

We will prove that the mean $\mathcal{M}_2(a, b\, ; p_1, p_2)$ even achieves a continuous refinement of classical inequality , $A_2(a, b) := \dfrac{a+b}{2} \leq \sqrt{\dfrac{a^2 + b^2}{2}} =: P_2(a, b)$, in the sense that it can take all values between $A_2(a, b)$ and $P_2(a, b)$ when the weights p_1 , p_2 vary freely in the range [0, 1] .

The *symmetry* and *homogeneity* of the mean $\mathcal{M}_2(a, b\, ; p_1, p_2)$ are also evident.

The notion of *continuous refinement* of an inequality was introduced in [6] .

For the complete study of the mean $\mathcal{M}_2(a, b\, ; p_1, p_2)$ - for convenience , we will take one from the two weights as variable - and we will associate a function to this mean .

4. *Proposition*

If $a, b \geq 0$ and $x \in [0,1]$, then the function $\mathbf{M}_2 : [0,1] \longrightarrow \mathbb{R}_{\geq 0}$, defined by

$$\mathrm{M}_2(x) := \frac{\sqrt{a^2 x + b^2(1-x)} + \sqrt{b^2 x + a^2(1-x)}}{2} ,$$

a) is continuous;
b) is increasing on $[0, ½]$ and decreasing on $[½, 1]$.

Proof

a) - obviously (operations and compositions of continuous functions) .
b) After elementary calculations, we have :

$$\mathrm{M}_2'(x) = \frac{\left(a^2-b^2\right)^2(1-2x)}{4\cdot\sqrt{\left[a^2x+b^2(1-x)\right]\cdot\left[b^2x+a^2(1-x)\right]}\cdot\left(\sqrt{a^2x+b^2(1-x)} + \sqrt{b^2x+a^2(1-x)}\right)} , \tag{8}$$

so the function $\mathbf{M}_2$ is increasing on $[0, ½]$ and decreasing on $[½, 1]$.

Moreover, we have : $\mathbf{M}_2(0) = \mathbf{M}_2(1) = \dfrac{a+b}{2}$, $\mathbf{M}_2(½) = \sqrt{\dfrac{a^2+b^2}{2}}$,

which reconfirms the results from *Proposition* 2 .

5. *Remark* The connection of the function $\mathbf{M}_2$ with the previously entered mean is given by the relation ,

$$\mathbf{M}_2(x) = \mathcal{M}_2(a, b; x, 1-x) , \quad x \in [0,1] , \tag{9}$$

which also denotes the mean character of the numbers a, b of the expression $\mathbf{M}_2(x)$.

6. *Two applications*

If in *Proposition* 2 we take $p_1 = sin^2\alpha$, $p_2 = cos^2\alpha$, for which we obviously have $p_1 + p_2 = 1$, we get the double inequality ,

$$\frac{a+b}{2} \leq \frac{\sqrt{a^2 sin^2\alpha + b^2 cos^2\alpha} + \sqrt{a^2 cos^2\alpha + b^2 sin^2\alpha}}{2} \leq \sqrt{\frac{a^2+b^2}{2}} . \tag{10}$$

Also, if in relation (2) we take $p_1 = 4/9$, $p_2 = 5/9$, we obtain ,

$$\frac{a+b}{2} \leq \frac{\sqrt{4a^2+5b^2} + \sqrt{4b^2+5a^2}}{6} \leq \sqrt{\frac{a^2+b^2}{2}} . \tag{11}$$

B. *The n-ary case*

The previously introduced $\mathcal{M}_2$ mean can be generalized in several ways, of which we would like to highlight here the following *three types of means* :

$$\mathcal{M}_{2,n}(a, b; p_1, p_2) = \frac{\sqrt[n]{p_1 a^n + p_2 b^n} + \sqrt[n]{p_1 b^n + p_2 a^n}}{2} , \tag{12}$$

where $a, b \in \mathbb{R}_{\geq 0}$ and the weights $p_1, p_2 \in [0,1]$, with $p_1 + p_2 = 1$.

$$\mathcal{M}_n([a],[p]) := \frac{1}{n}\cdot\left(\sqrt{p_1a_1^2+p_2a_2^2+\ldots+p_na_n^2}+\sqrt{p_1a_2^2+p_2a_3^2+\ldots+p_na_1^2}+\ldots+\sqrt{p_1a_n^2+p_2a_1^2+\ldots+p_na_{n-1}^2}\right), \quad (13)$$

where $[a] = (a_1, a_2, \ldots, a_n)$, $a_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$ and $[p] = (p_1, p_2, \ldots, p_n)$, $p_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$, with $p_1 + p_2 + \ldots + p_n = 1$.

$$\mathcal{M}_{n,m}([a],[p]) := \frac{1}{n}\cdot\left(\sqrt[m]{p_1a_1^m+p_2a_2^m+\ldots+p_na_n^m}+\sqrt[m]{p_1a_2^m+p_2a_3^m+\ldots+p_na_1^m}+\ldots+\sqrt[m]{p_1a_n^m+p_2a_1^m+\ldots+p_na_{n-1}^m}\right), \quad (14)$$

where $[a] = (a_1, a_2, \ldots, a_n)$, $a_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$ and $[p] = (p_1, p_2, \ldots, p_n)$, $p_k \in \mathbb{R}_{>0}$, $k = \overline{1,n}$, with $p_1 + p_2 + \ldots + p_n = 1$.

As obviously the means from relations (12) and (13) are particular cases of the mean from relation (14), it is sufficient to study the mean $\mathcal{M}_{n,m}([a],[p])$. For this we have the following,

7. *Theorem* , (see also , [7]) ,

If $[a] = (a_1, a_2, \ldots, a_n)$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}_{\geq 2}$, $a_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$ and $[p] = (p_1, p_2, \ldots, p_n)$, $p_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$, with $p_1 + p_2 + \ldots + p_n = 1$, then the following double inequality holds,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \leq \mathcal{M}_{n,m}([a],[p]) \leq \sqrt[m]{\frac{a_1^m + a_2^m + \ldots + a_n^m}{n}} \quad . \quad (15)$$

Proof

With the notation : $\mathbf{R}_1 = \sqrt[m]{p_1a_1^m + p_2a_2^m + \ldots + p_na_n^m}$, $\mathbf{R}_2 = \sqrt[m]{p_1a_2^m + p_2a_3^m + \ldots + p_na_1^m}$, ..., $\mathbf{R}_n = \sqrt[m]{p_1a_n^m + p_2a_1^m + \ldots + p_na_{n-1}^m}$, we obviously have the following relationship:

$$\mathbf{R}_1^m + \mathbf{R}_2^m + \ldots + \mathbf{R}_n^m = (p_1a_1^m + p_2a_2^m + \ldots + p_na_n^m) + (p_1a_1^m + p_2a_2^m + \ldots + p_na_n^m) + \ldots$$
$$\ldots + (p_1a_1^m + p_2a_2^m + \ldots + p_na_n^m) = (p_1 + p_2 + \ldots + p_n)\cdot(a_1^m + a_2^m + \ldots + a_n^m) =$$
$$= a_1^m + a_2^m + \ldots + a_n^m \quad . \quad (16)$$

For both inequalities we will use Jensen's inequality :

⊳ if $f : I \subset \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is a concave function on I , then for any $x_k \in \mathbf{I}$ and for any $p_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$, with $p_1 + p_2 + \ldots + p_n = 1$, we have :

$$f(p_1x_1 + p_2x_2 + \ldots + p_nx_n) \geq p_1f(x_1) + p_2f(x_2) + \ldots + p_nf(x_n) . \quad (17)$$

The function $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$, $f(x) = \sqrt[m]{x}$, $m \in \mathbb{N}^*$ is concave on $\mathbb{R}_{\geq 0}$, so with *Jensen's weighted inequality* - with $x_k = a_k^m$, $k = \overline{1,n}$, and with weights $p_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$, so that $p_1 + p_2 + \ldots + p_n = 1$, we have :

$$\mathbf{R}_1 = \sqrt[m]{p_1a_1^m + p_2a_2^m + \ldots + p_na_n^m} \geq p_1\sqrt[m]{a_1^m} + p_2\sqrt[m]{a_2^m} + \ldots + p_n\sqrt[m]{a_n^m} = p_1a_1 + p_2a_2 + \ldots + p_na_n .$$

Hence , $\quad \mathbf{R}_1 \geq p_1a_1 + p_2a_2 + \ldots + p_na_n$, $\quad (18_1)$

and analogs : $\quad \mathbf{R}_2 \geq p_1a_2 + p_2a_3 + \ldots + p_na_1$, $\quad (18_2)$

. . . .

$$\mathbf{R}_n \geq p_1 a_n + p_2 a_1 + \ldots + p_n a_{n-1} \quad , \qquad (18_n)$$

By adding relations (18_1) - (18_n) we get ,

$$\mathbf{R}_1 + \mathbf{R}_2 + \ldots + \mathbf{R}_n \geq (p_1 + p_2 + \ldots + p_n)(a_1 + a_2 + \ldots + a_n) = a_1 + a_2 + \ldots + a_n \, ,$$

and dividing by n yields the inequality on the left side of (15) .

Using *Jensen's inequality* again , for the concave function , $f(x) = \sqrt[m]{x}$, $m \in \mathbb{N}^*$, with $p_k = 1/n$, $k = \overline{1,n}$, and with $x_k = \mathbf{R}_k^m$, that is, in the form

$$f\left(\frac{\mathbf{R}_1^m + \mathbf{R}_2^m + \ldots + \mathbf{R}_n^m}{n}\right) \geq \frac{f(\mathbf{R}_1^m) + f(\mathbf{R}_2^m) + \ldots + f(\mathbf{R}_n^m)}{n} \Leftrightarrow$$

$$\Leftrightarrow \sqrt[m]{\frac{\mathbf{R}_1^m + \mathbf{R}_2^m + \ldots + \mathbf{R}_n^m}{n}} \geq \frac{\sqrt[m]{\mathbf{R}_1^m} + \sqrt[m]{\mathbf{R}_2^m} + \ldots + \sqrt[m]{\mathbf{R}_n^m}}{n}$$

and taking into account the relation (16) , we get ,

$$\sqrt[m]{\frac{a_1^m + a_2^m + \ldots + a_n^m}{n}} \geq \frac{\mathbf{R}_1 + \mathbf{R}_2 + \ldots + \mathbf{R}_n}{n} \quad ,$$

that is, the inequality on the right-hand side of (15) . Through particularizations, the following frameworks are obtained for the means introduced by relations (12) and (13) :

8. *Corollary*

$$\frac{a + b}{2} \leq \mathcal{M}_{2,n}(a, b \,;\, p_1, p_2) \leq \sqrt[n]{\frac{a^n + b^n}{2}} \quad , \qquad (19)$$

where $a, b \in \mathbb{R}_{\geq 0}$ and the weights $p_1, p_2 \in [\,0, 1\,]$, with $p_1 + p_2 = 1$.

9. *Corollary*

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \leq \mathcal{M}_n([a],[p]) \leq \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \quad , \qquad (20)$$

where $[a] = (a_1, a_2, \ldots, a_n)$, $a_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$, and $[p] = (p_1, p_2, \ldots, p_n)$, $p_k \in \mathbb{R}_{\geq 0}$, $k = \overline{1,n}$, with $p_1 + p_2 + \ldots + p_n = 1$.

10. *Application* (*a problem of Klamkin*)

We will apply *Corollary 9* to solve the following problem of the late mathematician and famous problemist - *Murray Klamkin* , [5] :

- if a, b, c are positive constants, determine the maximum and minimum of the expression ,

$$\sqrt{a^2x^2 + b^2y^2 + c^2z^2} + \sqrt{a^2y^2 + b^2z^2 + c^2x^2} + \sqrt{a^2z^2 + b^2x^2 + c^2y^2} \quad ,$$

knowing that $x^2 + y^2 + z^2 = 1$.

Considering in the relationship (20) : $n = 3$; $p_1 = x^2$, $p_2 = y^2$, $p_3 = z^2$; $a_1 = a$, $a_2 = b$, $a_3 = c$, we get ,

$$\frac{a+b+c}{3} \le \frac{1}{3}\cdot\left(\sqrt{a^2x^2+b^2y^2+c^2z^2}+\sqrt{a^2y^2+b^2z^2+c^2x^2}+\sqrt{a^2z^2+b^2x^2+c^2y^2}\right) \le \sqrt{\frac{a^2+b^2+c^2}{3}} .$$

After multiplying by 3, it follows that the minimum of the expression in the statement of the problem is $a+b+c$ and the maximum of the expression is equal to $\sqrt{3(a^2+b^2+c^2)}$.

References :

[1.] **Beckenbach E.F.** & **Bellman R.** , „ ***Inequalities*** " , Springer–Verlag , Berlin–Heidelberg – New-York , 1961 .

[2.] **Bencze M.** , „***New Means , New Inequalities And Refinements*" , Octogon Mathematical Magazine ,** vol **9** , no 1A , April 2001 , (pp.46-104).

[3.] **Bullen P. S.** & **Mitrinović D. S.** & **Vasić P. M.**, „***Means and Their Inequalities***" , D. Reidel Publidshing Company, Dordrecht/Boston , 1988 .

[4.] **Hardy G. H.** & **Littlewood J. E.** & **Polya G.**, „***Inequalities***" , 2 nd ed , Cambridge University Press , England , Cambridge (reprinted) , 1988 .

[5.] **Klamkin M.** , „***Problem 1667*", Mathematics Magazine ,** vol **76** , no 2 , Apr. 2003 , p.151.

[6.] **Mărghidanu D.** , „ ***Two Trigonometrical Means that Produce Continuous Refinement of the Inequality of the Classic Means*** ", in « **OCTOGON Mathematical Magazine** » , Vol.**12** , No. **2.A**., pp.664-667 , October , **2004** .

[7] **Mărghidanu Dorin** , ***Proposed problem* , (*A refinement of the inequality between arithmetic mean and power mean*)** **Mathematical Inequalities** , 31 August **2023** , https://www.facebook.com/photo/?fbid=6984183484974033&set=gm.3563705837250785&idorvanity=1486244404996949

[8] **Mitrinović D. S.** (in cooperation with **Vasić P. M.**) , „***Analytic Inequalities*** ", *Springer – – Verlag* , Band 165 , Berlin , 1970 .

[9] **Mitrinović D. S.** , **Pecarić J. E.** , **Fink A. M.**, „***Classical and New Inequalities in Analysis*** " , *Kluwer Acad. Press.* , 1993 .