An alternating Series involving Trigamma function

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Abstract: In this paper, we revive and bring to light the alternating square version of the trigamma series of Cornel Ioan Valean

$\sum_{n=1}^{\infty}(-1)^{n-1}(\psi^{(1)}(n))^2$ where $\psi^{(1)}(n)$ denotes Trigamma function

We evaluate this series by using a technique based on the computation of some special logarithmic and Di-logarithmic integrals.

let $S=\sum_{k=1}^{\infty}(-1)^{k-1}(\psi^{(1)}(k))^2$

Solution: As we know, the integral representation of trigamma function

$\psi^{(1)}(k)=\int_0^1 \frac{x^{k-1}\ln(x)}{1-x}dx$ ....(1)

using (1), we get:

$$\left(\psi^{(1)}(k)\right)^2=\int_0^1\int_0^1 \frac{\ln(y)\ln(x)}{(1-x)(1-y)}(xy)^{k-1}\,dxdy$$

Now, $\sum_{k=1}^{\infty}(-1)^{k-1}(\psi^{(1)}(k))^2=\int_0^1\int_0^1 \frac{\ln(y)\ln(x)}{(1-x)(1-y)}dxdy\sum_{k=1}^{\infty}(-xy)^{k-1}$

$=\int_0^1\int_0^1 \frac{\ln(y)\ln(x)}{(1-x)(1-y)(1+xy)}dxdy=\int_0^1\frac{\ln(x)}{1-x}\left(\int_0^1\frac{\ln(y)}{(1-y)(1+xy)}dy\right)dx$ ....(2)

Let $A=\int_0^1\frac{\ln(y)}{(1-y)(1+xy)}dy=\frac{x}{x+1}\int_0^1\frac{\ln(y)}{1+xy}dy+\frac{1}{1+x}\int_0^1\frac{\ln(y)}{1-y}dy$

$=\frac{x}{x+1}\sum_{n=1}^{\infty}(-x)^{n-1}\int_0^1 y^{n-1}\ln(y)\,dy+\frac{1}{x+1}\sum_{n=1}^{\infty}\int_0^1 y^{n-1}\ln(y)\,dy$

$=\frac{x}{x+1}\sum_{n=1}^{\infty}(-x)^{n-1}\left(\frac{-1}{n^2}\right)+\frac{1}{x+1}\sum_{n=1}^{\infty}\left(\frac{-1}{n^2}\right)$

$=\frac{1}{x+1}\sum_{n=1}^{\infty}\frac{(-x)^n}{n^2}-\frac{\xi(2)}{x+1}=\frac{Li_2(-x)}{x+1}-\frac{\xi(2)}{x+1}=\frac{1}{1+x}(Li_2(-x)-\xi(2))$ ....(3)

Using (3) in (2), we get,

$S=\int_0^1\frac{\ln(x)}{1-x}\left(\frac{1}{1+x}(Li_2(-x)-\xi(2))\right)dx=\int_0^1\frac{\ln(x)Li_2(-x)}{1-x^2}dx-\xi(2)\int_0^1\frac{\ln(x)}{1-x^2}dx$ ....(4)

Let $B=\int_0^1\frac{\ln(x)}{1-x^2}dx=\frac{1}{2}\int_0^1\frac{\ln(x)}{1-x}dx+\frac{1}{2}\int_0^1\frac{\ln(x)}{1+x}dx$

$=\frac{1}{2}\sum_{n=1}^{\infty}\int_0^1 x^{n-1}\ln(x)dx+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{n-1}\ln(x)dx$

$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{-1}{n^2}\right)+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{-1}{n^2}\right)$

$= \frac{-\xi(2)}{2}\frac{-\xi(2)}{4} = \frac{-3\xi(2)}{4} = -\frac{\pi^2}{8}$ ....(5)

Let $C = \int_0^1 \frac{\ln(x)Li_2(-x)}{1-x^2}dx = \frac{1}{2}\int_0^1 \frac{\ln(x)Li_2(-x)}{1+x}dx + \frac{1}{2}\int_0^1 \frac{\ln(x)Li_2(-x)}{1-x}dx$

$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n H_n^{(2)} \int_0^1 x^n \ln(x)\, dx + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^1 \frac{x^n \ln(x)}{1-x}dx$

$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{n^2} - H_n^{(2)}\right)\int_0^1 x^{n-1}\ln(x)\, dx + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(H_n^{(2)} - \xi(2)\right)$

$= \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n \left(\frac{1}{n^2} - H_n^{(2)}\right)\left(\frac{-1}{n^2}\right) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(H_n^{(2)} - \xi(2)\right)$

$= \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2} - \frac{1}{2}Li_4(-1) + \frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2} - \frac{1}{2}\xi(2)Li_2(-1)$

$=\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2} + \frac{17\pi^4}{1440}$

we have $\sum_{n=1}^{\infty}\frac{(-1)^n H_n^{(2)}}{n^2} = \frac{51\pi^4}{1440} - \frac{7}{2}\ln(2)\,\xi(3) + \frac{\pi^2}{6}\ln^2(2) - \frac{1}{6}\ln^4(2) - 4Li_4\left(\frac{1}{2}\right)$

then $C = \frac{17\pi^4}{360} - \frac{7}{2}\ln(2)\,\xi(3) + \frac{\pi^2}{6}\ln^2(2) - \frac{1}{6}\ln^4(2) - 4Li_4\left(\frac{1}{2}\right)$ ....(6)

Plugging (5) and (6) in (4), we get

$S = \frac{49\pi^4}{720} - \frac{7}{2}\ln(2)\,\xi(3) + \frac{\pi^2}{6}\ln^2(2) - \frac{1}{6}\ln^4(2) - 4Li_4\left(\frac{1}{2}\right)$

References:

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