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ABOUT AN INEQUALITY BY MARIN CHIRCIU FROM R.M.M.-43

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Here we present 7 methods to solve the problem J.2491 from R.M.M. – 43, Winter Edition, 2024, p. 84.

J.2491.

$$\frac{a}{7a+b+c}+\frac{b}{a+7b+c}+\frac{c}{a+b+7c}\le\frac{1}{3}\quad(1)$$

Inequality (1) can also be written in the form

$$\frac{1}{7}-\frac{a}{7a+b+c}+\frac{1}{7}-\frac{b}{a+7b+c}+\frac{1}{7}-\frac{c}{a+b+7c}\ge\frac{3}{7}-\frac{1}{3}$$

$$\frac{b+c}{7a+b+c}+\frac{c+a}{a+7b+c}+\frac{a+b}{a+b+7c}\ge\frac{2}{3}\quad(2)$$

SOLUTION 1. (REMOVE DENOMINATORS)

$3a(a+7b+c)(a+b+7c)+3b(a+b+7c)(7a+b+c)+3c(7a+b+c)(a+7b+c)\ge(7a+b+c)(a+7b+c)(a+b+7c)$

$$4(a^3+b^3+c^3)+12(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2)\ge 84abc\quad(3)$$

The inequality (3) yields by $AM-GM$: $a^3+b^3+c^3\ge 3abc, a^2b+ab^2+b^2c+bc^2+c^2a+ca^2\ge 6abc$.

SOLUTION 2. (BREAKING)

The inequality

$$\frac{a}{7a+b+c}\le\frac{5a+2b+2c}{27(a+b+c)}\qquad(4)$$

can be written as

$$8a^2-8a(b+c)+2(b+c)^2\ge 0\Leftrightarrow 2\big(2a-(b+c)\big)^2\ge 0.$$

Using the inequality (4) we obtain

$$\frac{a}{7a+b+c}+\frac{b}{a+7b+c}+\frac{c}{a+b+7c}\le$$

$$\leq \frac{5a+2b+2c}{27(a+b+c)}+\frac{5b+2c+2a}{27(a+b+c)}+\frac{5c+2a+2b}{27(a+b+c)} \leq \frac{9(a+b+c)}{27(a+b+c)}=\frac{1}{3}$$

SOLUTION 3. (HORNER BREAK)

Due to the homogeneity we can assume that $a+b+c=1$.
The inequality (2) becomes

$$\frac{a}{6a+1}+\frac{b}{6b+1}+\frac{c}{6c+1} \leq \frac{1}{3} \quad (5)$$

We want to determine m, n such that the inequality

$$\frac{x}{6x+1} \leq mx+n \quad (6)$$

To be true for any $x>0$. The inequality (6) is equivalent to

$$6mx^2+(m+6n-1)x+n \geq 0 \quad (7)$$

Bearing in mind that in the inequality (5) we have equality for $a=b=c=\frac{1}{3}$, must as the left side of the relation (7) to admit the double root $\frac{1}{3}$.
Using Horner's scheme we have

	$6m$	$m+6n-1$	n
$\frac{1}{3}$	$6m$	$3m+6n-1$	$m+3n-\frac{1}{3}$
$\frac{1}{3}$	$6m$	$5m+6n-1$	

By relations $m+3n-\frac{1}{3}=0, 5m+6n-1=0$ yields $m=\frac{1}{9}, n=\frac{2}{27}$. The inequality (7) becomes $2(3x-1)^2 \geq 0$.

Writing the inequality (6) for a, b, c we obtain

$$\frac{a}{6a+1}+\frac{b}{6b+1}+\frac{c}{6c+1} \leq \frac{3a+2}{27}+\frac{3a+2}{27}+\frac{3c+2}{27}=\frac{3(a+b+c)+6}{27}=\frac{1}{3}.$$

SOLUTION 4. (TANGENT METHOD)

Another method to determine m, n such that the inequality (6) is true for any $x>0$. For $x=\frac{1}{3}$ we obtain $\frac{m}{3}+n=\frac{1}{3}$. By derivation yields that $\frac{1}{(6x+1)^2}=m$; for $x=\frac{1}{3}$ we have $m=\frac{1}{9}$. For $m=\frac{1}{9}, n=\frac{2}{27}$, the inequality (6) becomes $2(3x-1)^2 \geq 0$, evidently true.

SOLUTION 5. (BERGSTRÖM)

Using the form by (2) and applying Bergström's inequality, we obtain

$$\frac{b+c}{7a+b+c}+\frac{c+a}{a+7b+c}+\frac{a+b}{a+b+7c}=$$

$$=\frac{(b+c)^2}{(b+c)(7a+b+c)}+\frac{(c+a)^2}{(c+a)(a+7b+c)}+\frac{(a+b)^2}{(a+b)(a+b+7c)}\geq$$

$$\geq\frac{(a+b+b+c+c+a)^2}{(a+b)(a+b+7c)+(b+c)(b+c+7a)+(c+a)(c+a+7b)}.$$

It remains to prove that

$$6(a+b+c)^2\geq(a+b)(a+b+7c)+(b+c)(b+c+7a)+(c+a)(c+a+7b)$$

$$4(a^2+b^2+c^2)\geq 4(ab+bc+ca),$$

Which is a well-known inequality.

SOLUTION 6. (SUBSTITUTIONS)

We denote $x=7a+b+c, y=7b+c+a, z=7c+a+b$. Solving this system, we get $a=\frac{8x-y-z}{54}, b=\frac{8y-z-x}{54}, c=\frac{8z-x-y}{54}$. The inequality (1) becomes

$$\frac{8x-y-z}{54x}+\frac{8y-z-x}{54y}+\frac{8z-x-y}{54z}\leq\frac{1}{3}$$

$$24-\left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\right)\leq 18$$

$$\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{z}{x}+\frac{x}{z}\geq 6,$$

which yields by $\frac{x}{y}+\frac{y}{x}\geq 2$.

SOLUTION 7. (DELIGATION)

We have

$$\frac{a}{7a+b+c}+\frac{b}{a+7b+c}+\frac{c}{a+b+7c}-\frac{1}{3}=$$

$$=\frac{a}{7a+b+c}-\frac{1}{9}+\frac{b}{a+7b+c}-\frac{1}{9}+\frac{c}{a+b+7c}-\frac{1}{9}=$$

$$=\frac{2a-b-c}{9(7a+b+c)}+\frac{2b-c-a}{9(a+7b+c)}+\frac{2c-a-b}{9(a+b+7c)}=$$

$$=\frac{a-b}{9(7a+b+c)}+\frac{a-c}{9(7a+b+c)}+\frac{b-c}{9(a+7b+c)}+\frac{b-a}{9(a+7b+c)}+\frac{c-a}{9(a+b+7c)}+\frac{c-b}{9(a+b+7c)}=$$

$$\frac{a-b}{9}\left(\frac{1}{7a+b+c}-\frac{1}{a+7b+c}\right)+\frac{b-c}{9}\left(\frac{1}{a+7b+c}-\frac{1}{a+b+7c}\right)+\frac{c-a}{9}\left(\frac{1}{a+b+7c}-\frac{1}{7a+b+c}\right)=$$

$$=-\frac{6}{9}\left(\frac{(a-b)^2}{(7a+b+c)(a+7b+c)}+\frac{(b-c)^2}{(a+7b+c)(a+b+7c)}+\frac{(c-a)^2}{(a+b+7c)(7a+b+c)}\right)\leq 0.$$

The equality occurs iff $a=b=c$.

REFERENCE:

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