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SOME INEQUALITIES SOLVED BY BW METHOD

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Abstract. This paper presents some inequalities solved by Buffalo Way (BW) method.
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I was going through an "article" on the "Buffalo Way", where the author said that one should *NEVER* use the Buffalo Way for proving inequalities in actual real-time contests as it is "highly inelengant". What is the reason behind this notion ? In Mathematics, there are a whole lot of ways to attempt a given question. If the BW provides a proof for some inequality, then why it is given the downvote ?[1]
As an answer to this question we will solve some inequalities by ''*Buffalo Way*'' (BW) method.

APPLICATION I. In [2], [3] and [5] was proved on many pages the following inequality:

If $x, y, z, t \geq 0$**, then:**

$$(x+y+z+t)(xyz+xyt+xzt+yzt) \geq (-x+y+z+t)(x-y+z+t)(x+y-z+t)(x+y+z-t).$$

Solution: Here we proved the inequality in few lines ! We use ''Buffalo Way'' (**BW**) method – see also [5]. We make substitutions:

(1) $x = \min\{x, y, z, t\},\ y = x+u, z = x+v, t = x+w$, wher $u, v, w \geq 0$.

The inequality to prove becomes:

$$(3\sum u^2 - 2\sum uv)x^2 + \left(4\sum u^3 - 2\sum uv(u+v) + 2uvw\right)x + \sum u^4 - 2\sum u^2v^2 + uvw(u+v+w) \geq 0.$$

We denote: $A = 3\sum u^2 - 2\sum uv,\ B = 4\sum u^3 - 2\sum uv(u+v) + 2uvw,$

$$C = \sum u^4 - 2\sum u^2v^2 + uvw(u+v+w);\ A = \sum u^2 + \sum (u-v)^2 \geq 0.$$

Since $\sum u^3+3uvw-\sum uv(u+v)\geq 0$ (*Schur*) and $\sum\frac{u+v}{w}=\sum\left(\frac{u}{v}+\frac{v}{u}\right)\geq 6>5\Rightarrow$

$\Rightarrow\sum uv(u+v)-5uvw\geq 0$, we obtain

$$B=4(\sum u^3+3uvw-\sum uv(u+v))+2(\sum uv(u+v)-5uvw)>0.$$

By $\sum u^4+uvw(u+v+w)\overset{Schur}{\geq}\sum uv(u^2+v^2)\overset{MA-MG}{\geq}2\sum u^2v^2$ we deduce $C\geq 0$.

APPLICATION II. If $x,y,z,t\geq 0$**, then:**

$$4(x^3+y^3+z^3+t^3)+15(xyz+xyt+xzt+yzt)\geq(x+y+z+t)^3,$$

(see the inequality 3 from [4], pag. 271).

Solution: Using (1) the inequality becomes

$12x^3+9\sum uv\cdot x^2+6\sum uv\cdot x+3\left(\sum u^3-\sum(u+v)uv+3uvw\right)\geq 0$, which is true because

$\sum u^3-\sum uv(u+v)+3uvw\geq 0$, (*Schur*).

APPLICATION III. If $x,y,z,t\geq 0$ **şi** $\lambda\geq\frac{11}{2}$ **, then**

$$\lambda(x^3+y^3+z^3+t^3)+(16-\lambda)(xyz+xyt+xzt+yzt)\geq(x+y+z+t)^3.$$

Solution: Using (1) the inequality becomes

$$(\lambda-4)(3\sum u^2-2\sum uv)x+$$
$$+(\lambda-1)\underbrace{(\sum u^3+3uvw-\sum uv(u+v))}_{\geq 0,(Schur)}+(\lambda-4)\sum uv(u+v)-(4\lambda-13)uvw\geq 0.$$

Since $(\lambda-4)\sum uv(u+v)-(4\lambda-13)uvw=uvw\left[(\lambda-4)\sum\left(\frac{u}{v}+\frac{v}{u}\right)+13-4\lambda\right]\geq$

$\geq uvw(6\lambda-24+13-4\lambda)=(2\lambda-11)uvw\geq 0$, the inequality is proved.

Remark. For $\lambda=\frac{11}{2}$ we obtain a problem proposed by *Vasile Cârtoaje* (M.S. 2006).

APPLICATION IV. If $x,y,z,t\geq 0$ **şi** $\alpha,\beta\in R$ **such that** $\alpha\geq 4,\alpha+\beta\geq 16,3\alpha+\beta\geq 27$ **, then**

$$\alpha(x^3+y^3+z^3+t^3)+\beta(xyz+xyt+xzt+yzt)\geq(x+y+z+t)^3.$$

(see the inequality 41 from [4], pag. 374).

Solution: Using (1) the inequality becomes

$$4(\alpha+\beta-16)x^3+3(\alpha+\beta-16)\sum u\cdot x^2+\left[(3\alpha-12)\sum u^2+(2\beta-24)\sum uv\right]\cdot x+$$

$$+(\alpha-1)\sum u^3+(\beta-6)uvw-3\sum uv(u+v)\geq 0.$$

Since $3\alpha+2\beta=\frac{6\alpha+4\beta}{2}=\frac{3\alpha+\beta+3(\alpha+\beta)}{2}\geq\frac{75}{2}\geq 36$, we deduce:

$$(3\alpha-12)\sum u^2+(2\beta-24)\sum uv\geq(3\alpha+2\beta-36)\sum uv\geq 0.$$

Since $\left[(\alpha-4)\sum\left(\frac{u}{v}+\frac{v}{u}\right)+\beta-3\alpha-3\right]\geq 6(\alpha-4)+\beta-3\alpha-3=3\alpha+\beta-27\geq 0$, we obtain

$$(\alpha-1)\sum u^3+(\beta-6)uvw-3\sum uv(u+v)=(\alpha-1)\underbrace{(\sum u^3+3uvw-\sum uv(u+v))}_{\geq 0,(Schur)}-$$

$$-(3\alpha-3)uvw+(\alpha-1)\sum uv(u+v)+(\beta-6)uvw-3\sum uv(u+v)=$$

$$=(\alpha-1)\underbrace{(\sum u^3+3uvw-\sum uv(u+v))}_{\geq 0,(Schur)}+uvw\left[(\alpha-4)\sum\left(\frac{u}{v}+\frac{v}{u}\right)+\beta-6-3\alpha+3\right]\geq 0.$$

Hence, from above yields the given inequality.On **BW** method see also [6].

References:

[1] https://math.stackexchange.com/questions/2120812/why-is-the-buffalo-way-considered-inelegant

[2] M. Cucoaneş, M. Drăgan – *Asupra unei inegalităţi algebrice*, R.M.T., Nr. 3 (2018),

pag. 15-16.

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[4] V. Cîrtoaje – *Algebraic Inequalities (Old and New Methods)*, GIL Publishing House, Zalău, Romania, 2006.

[5] https://artofproblemsolving.com/community/c6h605279

[6] https://artofproblemsolving.com/community/c6h522084