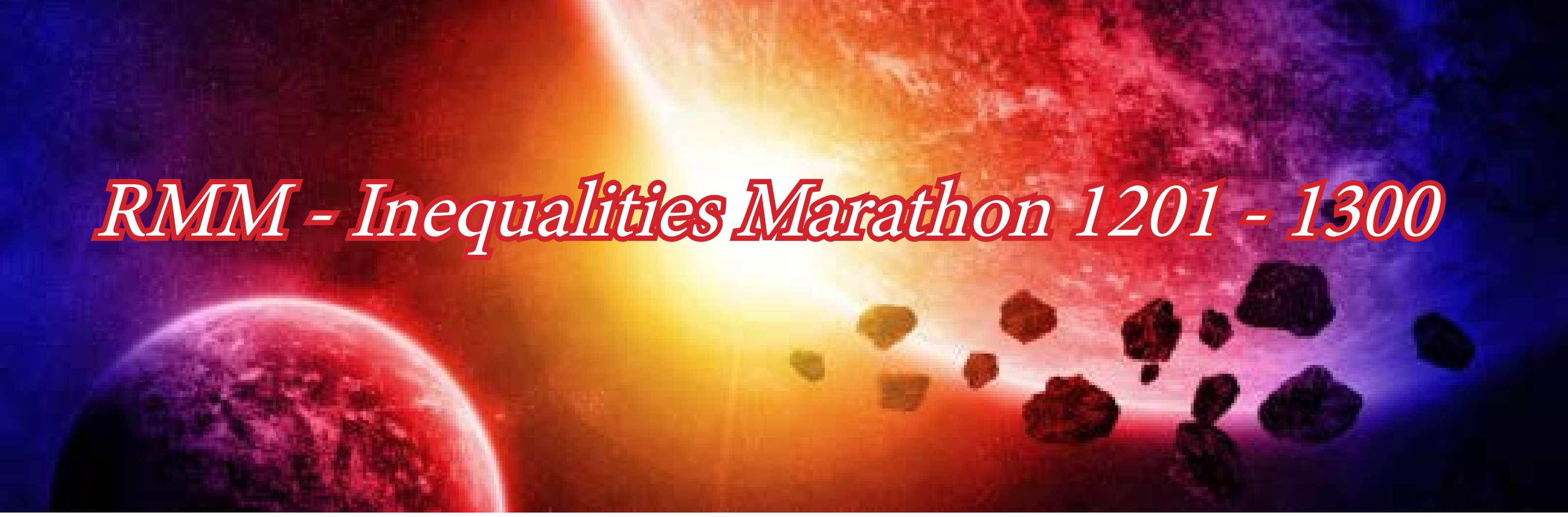

RMM - Inequalities Marathon 1201 - 1300

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1201. Prove whithout any software:

$$9\cdot 4^{\sqrt{5}}\cdot 5^{\sqrt{6}}\cdot 6^{\sqrt{7}} < 25\cdot 2^{\sqrt{3}}\cdot 3^{\sqrt{2}}\cdot 6^{\sqrt{5}}\cdot 7^{\sqrt{6}}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Pham Duc Nam-Vietnam

Let be $f(x)=\frac{lnx}{\sqrt{x}}, f'(x)=\frac{2-lnx}{\sqrt{x}}>0, \forall x\in(1,e^2)\to f$ increasing on $(1,e^2)$

$$4<5\Rightarrow f(4)<f(5)\Rightarrow\frac{ln4}{\sqrt{4}}<\frac{ln5}{\sqrt{5}}\Rightarrow\sqrt{5}ln4<\sqrt{4}ln5\Rightarrow$$
$$ln4^{\sqrt{5}}<ln5^{\sqrt{4}}\Rightarrow 4^{\sqrt{5}}<5^{\sqrt{4}}$$

Analogous:

$$5^{\sqrt{6}}<6^{\sqrt{5}}, 6^{\sqrt{7}}<7^{\sqrt{6}}\Rightarrow 4^{\sqrt{5}}\cdot 5^{\sqrt{6}}\cdot 6^{\sqrt{7}}<5^{\sqrt{4}}\cdot 6^{\sqrt{5}}\cdot 7^{\sqrt{6}}\Rightarrow$$
$$\frac{4^{\sqrt{5}}\cdot 5^{\sqrt{6}}\cdot 6^{\sqrt{7}}}{5^{\sqrt{4}}\cdot 6^{\sqrt{5}}\cdot 7^{\sqrt{6}}}<1$$

Remains to prove:

$$9<2^{\sqrt{3}}\cdot 3^{\sqrt{2}}$$
$$\frac{5}{3}<\sqrt{3}\text{ because }25<27\Rightarrow 2^{\frac{5}{3}}<2^{\sqrt{3}}$$
$$\frac{4}{3}<\sqrt{2}\text{ because }16<18\Rightarrow 3^{\frac{4}{3}}<3^{\sqrt{2}}$$
$$2^{\frac{5}{3}}\cdot 3^{\frac{4}{3}}<2^{\sqrt{3}}\cdot 3^{\sqrt{2}}$$
$$9=729^{\frac{1}{3}}<2592^{\frac{1}{3}}=\left(2^5\cdot 3^4\right)^{\frac{1}{3}}=2^{\frac{5}{3}}\cdot 3^{\frac{4}{3}}<2^{\sqrt{3}}\cdot 3^{\sqrt{2}}$$

Solution 2 by Le Thu-Vietnam

Let be $f(x)=x^{\frac{1}{x}}$

$$e<x<y\Rightarrow f(x)>f(y)$$
$$0<x<y<e\Rightarrow f(x)<f(y)$$
$$f^{xy}(x)<f^{xy}(y)\Rightarrow x^y<y^x\Rightarrow x^{\frac{\sqrt{y}}{2}}<y^{\frac{\sqrt{x}}{2}}\Rightarrow$$

$x^{\sqrt{y}}<y^{\sqrt{x}}$ for $0<x<y<e^2\approx 7.3$

$$25\cdot 2^{\sqrt{3}}\cdot 3^{\sqrt{2}}\cdot 6^{\sqrt{5}}\cdot 7^{\sqrt{6}}>5^2\cdot 2^{2\sqrt{3}}\cdot 5^{\sqrt{6}}\cdot 6^{\sqrt{7}}>9\cdot 4^{\sqrt{5}}\cdot 5^{\sqrt{6}}\cdot 6^{\sqrt{7}}$$

since $2^{\sqrt{\frac{3}{5}}}>\left(\sqrt{\frac{3}{5}}\right)^2\Leftrightarrow 4^{\sqrt{3}}\cdot 25>4^{\sqrt{5}}\cdot 9$

Solution 3 by Hikmat Mammadov-Azerbaijan

Let be:

$f(x)=\frac{lnx}{\sqrt{x}}, f'(x)=\frac{2-lnx}{\sqrt{x}}>0, \forall x\in(1,e^2)\to f$ increasing on $(1,e^2)$

$$2<3\Rightarrow f(2)<f(3)\Rightarrow\frac{ln2}{\sqrt{2}}<\frac{ln3}{\sqrt{3}}\Rightarrow\sqrt{3}ln2<\sqrt{2}ln3\Rightarrow$$
$$ln2^{\sqrt{3}}<ln3^2\Rightarrow 2^{\sqrt{3}}<3^{\sqrt{2}}$$

Analogous:

$$3^{\sqrt{4}} < 4^{\sqrt{3}}, 4^{\sqrt{5}} < 5^{\sqrt{4}}, 5^{\sqrt{6}} < 6^{\sqrt{5}}, 6^{\sqrt{7}} < 7^{\sqrt{6}}$$

$$2^{\sqrt{3}} \cdot 3^{\sqrt{4}} \cdot 4^{\sqrt{5}} \cdot 5^{\sqrt{6}} \cdot 6^{\sqrt{7}} < 3^{\sqrt{2}} \cdot 4^{\sqrt{3}} \cdot 5^{\sqrt{4}} \cdot 6^{\sqrt{5}} \cdot 7^{\sqrt{6}}$$

$$2^{\sqrt{3}} \cdot 9 \cdot 4^{\sqrt{5}} \cdot 5^{\sqrt{6}} \cdot 6^{\sqrt{7}} < 3^{\sqrt{2}} \cdot 2^{2\sqrt{3}} \cdot 25 \cdot 6^{\sqrt{5}} \cdot 7^{\sqrt{6}}$$

$$9 \cdot 4^{\sqrt{5}} \cdot 5^{\sqrt{6}} \cdot 6^{\sqrt{7}} < 3^{\sqrt{2}} \cdot 2^{\sqrt{3}} \cdot 25 \cdot 6^{\sqrt{5}} \cdot 7^{\sqrt{6}}$$

$$9 \cdot 4^{\sqrt{5}} \cdot 5^{\sqrt{6}} \cdot 6^{\sqrt{7}} < 25 \cdot 2^{\sqrt{3}} \cdot 3^{\sqrt{2}} \cdot 6^{\sqrt{5}} \cdot 7^{\sqrt{6}}$$

1202. If $a \in \left(0, \frac{\pi}{2}\right)$ then:

$$\frac{\sin 2a}{\sin a + \cos a} + \frac{1}{\sqrt{2}} \geq \sqrt{\frac{\sin 2a}{2}} + \frac{\sin a + \cos a}{2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Christos Tsifakis-Greece

$$\frac{\sin 2a}{\sin a + \cos a} + \frac{1}{\sqrt{2}} \geq \sqrt{\frac{\sin 2a}{2}} + \frac{\sin a + \cos a}{2}$$

$$\frac{1}{\sqrt{2}} \geq \sqrt{\frac{\sin 2a}{2}} + \frac{\sin a + \cos a}{2} - \frac{\sin 2a}{\sin a + \cos a}$$

$$\frac{1}{\sqrt{2}} \geq \sqrt{\frac{\sin 2a}{2}} + \frac{(\sin a + \cos a)^2 - \sin 2a}{2(\sin a + \cos a)}$$

$$\frac{1}{\sqrt{2}} \geq \sqrt{\frac{\sin 2a}{2}} + \frac{1 - \sin 2a}{2(\sin a + \cos a)}$$

$$\frac{1 - \sin 2a}{2(\sin a + \cos a)} \leq \frac{1 - \sqrt{\sin 2a}}{\sqrt{2}}, \qquad \frac{1 - \sin 2a}{2(\sin a + \cos a)} \leq \frac{1 - \sin 2a}{\sqrt{2}\left(1 + \sqrt{\sin 2a}\right)}$$

$$\frac{1}{2(\sin a + \cos a)} \leq \frac{1}{\sqrt{2}\left(1 + \sqrt{\sin 2a}\right)}$$

$$2(\sin a + \cos a) \geq \sqrt{2}\left(1 + \sqrt{\sin 2a}\right)$$

$$2(1 + \sin 2a) \geq 1 + \sin 2a + 2\sqrt{\sin 2a}$$

$$\sin 2a - 2\sqrt{\sin 2a} + 1 \geq 0$$

$$\left(1 - \sqrt{\sin 2a}\right)^2 \geq 0$$

Equality holds for $a = \frac{\pi}{4}$.

Solution 2 by Fayssal Abdelli-Bejaia-Algerie

Suppose by absurdum that:

$$\frac{\sin 2a}{\sin a+\cos a}+\frac{1}{\sqrt{2}}<\sqrt{\frac{\sin 2a}{2}}+\frac{\sin a+\cos a}{2}$$

$$\frac{1}{\sqrt{2}}<\sqrt{\frac{\sin 2a}{2}}+\frac{\sin a+\cos a}{2}-\frac{\sin 2a}{\sin a+\cos a}$$

$$\frac{1}{\sqrt{2}}<\sqrt{\frac{\sin 2a}{2}}+\frac{(\sin a+\cos a)^2-\sin 2a}{2(\sin a+\cos a)}$$

$$\frac{1}{\sqrt{2}}<\sqrt{\frac{\sin 2a}{2}}+\frac{1-\sin 2a}{2(\sin a+\cos a)}$$

$$\frac{1-\sin 2a}{2(\sin a+\cos a)}>\frac{1-\sqrt{\sin 2a}}{\sqrt{2}}$$

$$\frac{1-\sin 2a}{2(\sin a+\cos a)}>\frac{1-\sin 2a}{\sqrt{2}\left(1+\sqrt{\sin 2a}\right)}$$

$$\frac{1}{2(\sin a+\cos a)}>\frac{1}{\sqrt{2}\left(1+\sqrt{\sin 2a}\right)}$$

$$2(\sin a+\cos a)<\sqrt{2}\left(1+\sqrt{\sin 2a}\right)$$

$$2(1+\sin 2a)<1+\sin 2a+2\sqrt{\sin 2a}$$

$$\sin 2a-2\sqrt{\sin 2a}+1<0$$

$$\left(1-\sqrt{\sin 2a}\right)^2<0.\text{False}$$

Equality holds for $a=\frac{\pi}{4}$.

Solution 3 by Ravi Prakash-New Delhi-India

Let $b=\sin a+\cos a, a\in\left(0,\frac{\pi}{2}\right)$.

$$1=\sin^2 b+\cos^2 b\le \sin b+\cos b=\sqrt{2}\sin\left(\frac{\pi}{4}+b\right)\le\sqrt{2}$$

$$1\le b\le\sqrt{2}\Rightarrow 1\le b^2\le 2$$

$$b^2=1+\sin 2a\Rightarrow \sin 2a=b^2-1$$

We have to prove:

$$\frac{b^2-1}{b}+\frac{1}{\sqrt{2}}\ge\sqrt{\frac{b^2-1}{2}}+\frac{b}{2}$$

$$b-\frac{1}{b}-\frac{b}{2}\ge\frac{1}{\sqrt{2}}\cdot\left(\sqrt{b^2-1}-1\right),\qquad \frac{b^2-2}{2b}\ge\frac{1}{\sqrt{2}}\cdot\frac{b^2-1-1}{\sqrt{b^2-1}+1}$$

$$\frac{1}{2b}\le\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{b^2-1}+1},\quad \sqrt{2}\cdot\left(\sqrt{b^2-1}+1\right)\le 2b$$

$$\sqrt{b^2-1}\le\sqrt{2}b-1$$

$$b^2-1\le 2b^2-2\sqrt{2}b+1$$
$$b^2-2\sqrt{2}b+2\ge 0$$
$$\left(b-\sqrt{2}\right)^2\ge 0$$

Equality holds for $a=\frac{\pi}{4}$.

Solution 4 by Tapas Das-India

$$x=sin2a\Rightarrow (sina+cosa)^2=1+x$$

$$\frac{sin2a}{sina+cosa}+\frac{1}{\sqrt{2}}\ge\sqrt{\frac{sin2a}{2}+\frac{sina+cosa}{2}}$$

$$\frac{x^2-1}{x}+\frac{1}{\sqrt{2}}\ge\sqrt{\frac{x^2-1}{2}+\frac{x}{2}},\quad x-\frac{1}{x}-x\ge\frac{1}{\sqrt{2}}\cdot\left(\sqrt{x^2-1}-1\right)$$

$$\frac{x^2-2}{2x}\ge\frac{1}{\sqrt{2}}\cdot\frac{x^2-1-1}{\sqrt{x^2-1}+1},\qquad \frac{1}{2x}\le\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{x^2-1}+1}$$

$$\sqrt{2}\cdot\left(\sqrt{x^2-1}+1\right)\le 2x$$

$$\sqrt{x^2-1}\le\sqrt{2}x-1,\qquad x^2-1\le 2x^2-2\sqrt{2}x+1$$

$$x^2-2\sqrt{2}x+2\ge 0,\qquad \left(x-\sqrt{2}\right)^2\ge 0$$

Equality holds for $a=\frac{\pi}{4}$.

1203.

$$x,y>0,x+y=2,a=\left[2log_2\frac{2^{256}}{\binom{256}{128}}\right],[*]-GIF$$

Prove that:

$$\sqrt{a+\frac{1}{x^2}}+\sqrt{a+\frac{1}{y^2}}\ge 6$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Hikmat Mammadov-Azerbaijan

$$e^x\ge x+1,x\in\mathbb{R}\Rightarrow e^{-\frac{1}{1024}}>1-\frac{1}{1024}=\frac{1023}{1024}\Rightarrow e^{\frac{1}{1024}}<\frac{1024}{1023}$$

$$\sqrt{2n\pi}\cdot n^n\cdot e^{-n}\cdot e^{\frac{1}{12n+1}}<n!<\sqrt{2n\pi}\cdot n^n\cdot e^{-n}\cdot e^{\frac{1}{12n}}$$

$$2n\pi\cdot n^{2n}\cdot e^{-2n}\cdot e^{\frac{2}{12n+1}}<(n!)^2<2n\pi\cdot n^{2n}\cdot e^{-2n}\cdot e^{\frac{1}{6n}}$$

$$\frac{1}{\sqrt{2n\cdot 2\pi}}\cdot\frac{1}{(2n)^{2n}}\cdot\frac{1}{e^{-2n}}\cdot\frac{1}{e^{\frac{1}{24n+1}}}\le\frac{1}{(2n)!}\le\frac{1}{\sqrt{2n\cdot 2\pi}}\cdot\frac{1}{(2n)^{2n}}\cdot\frac{1}{e^{-2n}}\cdot\frac{1}{e^{\frac{1}{24n}}}$$

$$\sqrt{\pi}\cdot\sqrt{n}\cdot\frac{1}{2^{2n}}\cdot e^{\frac{2}{12n+1}-\frac{1}{24n+1}}<\frac{(n!)^2}{(2n)!}<\sqrt{\pi}\cdot\sqrt{n}\cdot\frac{1}{2^{2n}}\cdot e^{\frac{1}{6n}-\frac{1}{24n}}<$$

$$<\frac{(128!)^2\cdot 2^{256}}{256!}<\sqrt{\pi}\cdot\sqrt{128}\cdot e^{\frac{1}{1024}}<\sqrt{\frac{22}{7}\cdot 128\cdot\frac{1024}{1023}}<16\sqrt{2}=2^{4.5}$$

$$4<\log_2\frac{2^{256}}{\binom{256}{128}}<4.5\Rightarrow 8<2\log_2\frac{2^{256}}{\binom{256}{128}}<9\Rightarrow a=8$$

$$\sqrt{a+\frac{1}{x^2}}+\sqrt{a+\frac{1}{y^2}}\overset{MINKOVSKI}{\geq}\sqrt{\left(\sqrt{a}+\sqrt{a}\right)^2+\left(\frac{1}{x}+\frac{1}{y}\right)^2}\overset{AM-HM}{\geq}$$

$$\geq\sqrt{\left(2\sqrt{a}\right)^2+\left(\frac{4}{x+y}\right)^2}=\sqrt{4a+\left(\frac{4}{2}\right)^2}=\sqrt{4\cdot 8+4}=6$$

Solution 2 by Pham Duc Nam-Vietnam

$$a=\left[2\log_2\frac{2^{256}}{\binom{256}{128}}\right]$$

$x\in(0,2)$. **We prove that:**

$$\sqrt{8+\frac{1}{x^2}}\geq\frac{10-x}{3}\ \ (equality\ for\ x=1)$$

$$\frac{8x^2+1}{x^2}\geq\left(\frac{10-x}{3}\right)^2\Leftrightarrow(x-1)^2(90-(x-9)^2)\geq 0$$

$$\sqrt{8+\frac{1}{x^2}}+\sqrt{8+\frac{1}{y^2}}\geq\frac{10-x}{3}+\frac{10-y}{3}=\frac{20-2}{3}=6$$

1204.

Let $m>1, n\in\mathbb{N}_{\geq 1}, k,i\in\overline{1,n}$ and $H_{n,k,m}=\sum_{i=1}^{n}\frac{1}{i^{km}}$. Prove that

$$\sum_{k=1}^{n}\sqrt[m]{H_{n,k,m}\cdot\left(\sum_{i=k}^{n}\sqrt[m-1]{\left(\binom{i}{k}\cdot i^k\right)^m}\right)^{m-1}}\geq 2(2^n-1)-n$$

Proposed by Khaled Abd Imouti-Syria, Sidi Abdallah Lemrabott-Mauritania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By Hölder's inequality, we have

$$H_{n,k,m}\cdot\left(\sum_{i=k}^{n}\sqrt[m-1]{\left(\binom{i}{k}\cdot i^k\right)^m}\right)^{m-1}\geq\left(\sum_{i=k}^{n}\frac{1}{i^{km}}\right)\cdot\left(\sum_{i=k}^{n}\sqrt[m-1]{\left(\binom{i}{k}\cdot i^k\right)^m}\right)^{m-1}$$

$$\geq\left(\sum_{i=k}^{n}\sqrt[m]{\frac{1}{i^{km}}\cdot\left(\sqrt[m-1]{\left(\binom{i}{k}.i^k\right)^m}\right)^{m-1}}\right)^m=\left(\sum_{i=k}^{n}\binom{i}{k}\right)^m.$$

Then

$$\sum_{k=1}^{n}\sqrt[m]{H_{n,k,m}\cdot\left(\sum_{i=k}^{n}\sqrt[m-1]{\left(\binom{i}{k}.i^k\right)^m}\right)^{m-1}}\geq\sum_{k=1}^{n}\sum_{i=k}^{n}\binom{i}{k}$$

$$=\sum_{k=1}^{n}\left(1+\sum_{i=k+1}^{n}\left(\binom{i+1}{k+1}-\binom{i}{k+1}\right)\right)$$

$$=n+\sum_{k=1}^{n}\left(\binom{n+1}{k+1}-\binom{k+1}{k+1}\right)=n+\left(2^{n+1}-\binom{n+1}{0}-\binom{n+1}{1}\right)-n$$

$$=2(2^n-1)-n,$$

as desired.

1205.

$$n\in\{1,2,\dots\}\ \Rightarrow\ \sqrt[n+1]{1+(n+1)\sqrt{\binom{2n}{n}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2}\right)}}>2$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Since }\binom{2n}{n}=\binom{n}{0}^2+\binom{n}{1}^2+\cdots+\binom{n}{n}^2,$$

$$\text{then by the Cauchy}-\text{Schwarz inequality, we have}$$

$$\sqrt{\binom{2n}{n}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2}\right)}=\sqrt{\sum_{k=0}^{n}\binom{n}{k}^2\cdot\sum_{k=0}^{n}\frac{1}{(k+1)^2}}\geq\sum_{k=0}^{n}\frac{\binom{n}{k}}{k+1}=$$

$$=\sum_{k=0}^{n}\frac{\binom{n+1}{k+1}}{n+1}=\frac{2^{n+1}-1}{n+1},$$

$$\text{with equality if }(k+1)\binom{n}{k}=\text{constant},\quad\forall k\in\{0,1,\dots,n\},$$

$$\text{which is not true for any }n\geq 1.\text{ Therefore}$$

$$\sqrt[n+1]{1+(n+1)\sqrt{\binom{2n}{n}\left(1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{(n+1)^2}\right)}}>2.$$

1206. If $n \in \mathbb{N}$, then prove that :

$$\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{4n}\right) < \left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{4(n+1)}\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$n \neq 0, n \in \mathbb{N} \Rightarrow n \geq 1 \text{ and } \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{4n}\right)$$

$$< \left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{4(n+1)}\right)$$

$$\Leftrightarrow \left(\frac{1+\frac{1}{n+1}}{1+\frac{1}{n}}\right)^n\left(\frac{n+2}{n+1}\right)\left(\frac{4n^2+4n+1}{4(n+1)}\right)\left(\frac{4n}{4n+1}\right) > 1$$

$$\Leftrightarrow \left(\frac{n(n+2)}{(n+1)^2}\right)^n \cdot \frac{n(n+2)(2n+1)^2}{(4n+1)(n+1)^2} \overset{(*)}{>} 1$$

$$\text{Now, } \left(\frac{n(n+2)}{(n+1)^2}\right)^n = \left(1+\frac{n(n+2)}{(n+1)^2}-1\right)^n = \left(1-\frac{1}{(n+1)^2}\right)^n \overset{\text{Bernoulli}}{\geq}$$

$$1-\frac{n}{(n+1)^2}\left(\because n+1>1 \Rightarrow 1 > \frac{1}{(n+1)^2} \Rightarrow -\frac{1}{(n+1)^2} > -1 \text{ and } n \geq 1\right)$$

$$= \frac{n^2+n+1}{(n+1)^2} \Rightarrow \text{LHS of } (*) \geq \frac{n^2+n+1}{(n+1)^2} \cdot \frac{n(n+2)(2n+1)^2}{(4n+1)(n+1)^2} \overset{?}{>} 1$$

$$\Leftrightarrow n(n+2)(n^2+n+1)(2n+1)^2 - (4n+1)(n+1)^4 \overset{?}{>} 1$$

$$\Leftrightarrow 4n^6+12n^5+8n^4-5n^3-11n^2-6n-1 \overset{?}{>} 1$$

$$\Leftrightarrow (n-1)(4n^5+16n^4+24n^3+19n^2+8n+2)+1 \overset{?}{>} 0 \rightarrow \text{true} \because n \geq 1 \Rightarrow (*)$$

$$\text{is true} \therefore \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{4n}\right) < \left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{4(n+1)}\right) \forall n \in \mathbb{N} \text{ (QED)}$$

1207. If $x, y, z, t > 0, xyzt = 1$ then:

$$(x+y+z+t)^2 \leq 2\left(xy+\frac{1}{xy}\right)\left(xz+\frac{1}{xz}\right)\left(xt+\frac{1}{xt}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z, t > 0$

$$(x+y+z+t)^2 \leq 2\left(xy+\frac{1}{xy}\right)\left(xz+\frac{1}{xz}\right)\left(xt+\frac{1}{xt}\right)$$

Iff $(x+y+z+t)^2 \le (xy+zt)(xz+yt)(xt+yz)$

Iff $x^2+y^2+z^2+t^2+2(xy+xz+xt+yz+yt+zt) \le$

$$\le 2\left(x^2+y^2+z^2+t^2+\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}\right)$$

Iff $2(xy+xz+xt+yz+yt+zt) \le x^2+y^2+z^2+t^2+2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2}\right)$ ok

Because $(x^2+y^2+z^2+t^2) \ge xy+yz+zt+tx$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2} \ge \frac{1}{xy}+\frac{1}{yz}+\frac{1}{zt}+\frac{1}{tx} = zt+xt+xy+yz$$

$$\frac{1}{x}+\frac{1}{y^2}+\frac{1}{z^2}+\frac{1}{t^2} \ge \frac{2}{xz}+\frac{2}{yt} = 2(yt+zx)$$

Therefore it is to be true

Solution 2 by Bui Hong Suc-Vietnam

$$xyzt = 1 \leftrightarrow \begin{cases} xt = \frac{1}{yz} \\ xz = \frac{1}{yt} \end{cases}$$

$$RHS = 2\left(xy+\frac{1}{xy}\right)\left(xz+\frac{1}{xz}\right)\left(xt+\frac{1}{xt}\right) = 2(xy+zt)(xz+yt)(xt+yz) =$$

$$= 2(x^2yz+xz^2t+xy^2t+yzt^2)(xt+yz)$$

$$= 2(x^2xyzt+x^2z^2t^2+x^2y^2t^2+xyztt^2+x^2y^2z^2+xyztz^2+xyzy^2t+y^2z^2t^2)$$

$$= 2\left(x^2+\frac{1}{y^2}+\frac{1}{z^2}+t^2+\frac{1}{t^2}+z^2+y^2+\frac{1}{x^2}\right)$$

$$= x^2+y^2+z^2+t^2+(x^2+y^2)+(z^2+t^2)+\left(\frac{1}{y^2}+\frac{1}{z^2}\right)+\left(\frac{1}{t^2}+\frac{1}{x^2}\right)+$$

$$+\left(\frac{1}{z^2}+\frac{1}{x^2}\right)+\left(\frac{1}{y^2}+\frac{1}{t^2}\right)$$

$$\overset{AGM}{\ge} x^2+y^2+z^2+t^2+2xy+2zt+2\cdot\frac{1}{yz}+2\cdot\frac{1}{tx}+2\cdot\frac{1}{zx}+2\cdot\frac{1}{yt}$$

$$= x^2+y^2+z^2+t^2+2xy+2zt+2xt+2yz+2yt+2xz$$

$$= (x+y+z+t)^2 = LHS$$

Hence: $(x+y+z+t)^2 \le 2\left(xy+\frac{1}{xy}\right)\left(xz+\frac{1}{xz}\right)\left(xt+\frac{1}{xt}\right)$

1208. **Prove without any software:**

$$\frac{\pi\sqrt{3}}{9} < \log 3 < \frac{\pi\sqrt{3}}{3}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Adrian Popa – Romania

$$\frac{\pi\sqrt{3}}{9} < \ln 3 < \frac{\pi\sqrt{3}}{3} \Rightarrow \frac{\pi}{9} \overset{?}{<} 3^{-\frac{1}{2}} \ln 3 < \frac{\pi}{3}$$

$$\frac{\pi}{9} < \ln 3^{\frac{1}{\sqrt{3}}} < \frac{\pi}{3}, \qquad \ln e^{\frac{\pi}{9}} < \ln 3^{\frac{1}{\sqrt{3}}} < \ln e^{\frac{\pi}{3}}$$

$$e^{\frac{\pi}{9}} < 3^{\frac{1}{\sqrt{3}}} < e^{\frac{\pi}{3}} |^3, \qquad e^{\frac{\pi}{3}} < 3^{\sqrt{3}} < e^{\pi}$$

$$\left.\begin{matrix} e < 3 \\ \frac{\pi}{3} < \sqrt{3} \end{matrix}\right\} \Rightarrow e^{\frac{\pi}{3}} < 3^{\sqrt{3}} \quad (1)$$

$$3^{\sqrt{3}} < \pi^e < e^{\pi} \quad (2)$$

From (1) and (2) $\Rightarrow \frac{\pi\sqrt{3}}{9} < \ln 3 < \frac{\pi\sqrt{3}}{3}$

It remains to prove that $e^e < e^{\pi} \Leftrightarrow e \ln \pi < \pi \ln e \Leftrightarrow \frac{\ln \pi}{\pi} < \frac{\ln e}{e}$

Let be the function $f: (0; +\infty) \to \mathbb{R}; f(x) = \frac{\ln x}{x}$

$$f'(x) = \frac{1 - \ln x}{x^2} = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$$

x	0 $\quad e \quad$ π $\quad +\infty$
$f'(x)$	$+ + + + 0 - - - - - - - - - - - -$
$f(x)$	$-\infty \nearrow \frac{1}{e} \searrow 0$

$$\Rightarrow f(e) > f(\pi) \Rightarrow \frac{\ln e}{e} > \frac{\ln \pi}{\pi} \Rightarrow \pi^e < e^{\pi}$$

Solution 2 by Jaegyuttasart Torsak-Thailand

Prove: $\frac{\pi\sqrt{3}}{9} < \ln 3 < \frac{\pi\sqrt{3}}{3}$ or $\frac{\pi}{3\sqrt{3}} < \ln 3 < \frac{\pi}{\sqrt{3}}$

1. $\frac{\pi}{3\sqrt{3}} \overset{?}{<} 1 \to \pi < 3\sqrt{3}$ True

$$2.\ 1 \overset{?}{<} \ln 3 \to e < 3 \text{ True}$$

$$3.\ \ln 3 = \int_0^2 \frac{dx}{1+x} \overset{CS}{<} \sqrt{\int_0^2 \frac{dx}{(1+x)^2} \cdot \int_0^2 dx} = \frac{2}{\sqrt{3}}$$

$$4.\ \frac{2}{\sqrt{3}} \overset{?}{<} \frac{\pi}{\sqrt{3}} \to 2 < \pi \text{ True}$$

$$1 \to 4;\ \frac{\pi}{3\sqrt{3}} < 1 < \ln 3 < \frac{2}{\sqrt{3}} < \frac{\pi}{\sqrt{3}}$$

1209.

If $0 < a_1 \le a_2 \le \dots \le a_n$ then

$$\sum_{i=1}^{n} \frac{1}{a_i} \le \frac{1}{a_1 a_n}\left(n(a_1 + a_n) - \sum_{i=1}^{n} a_i\right)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality is equivalent to

$$0 \le \sum_{i=1}^{n}\left[\frac{1}{a_1 a_n}\big((a_1 + a_n) - a_i\big) - \frac{1}{a_i}\right] \Leftrightarrow 0 \le \sum_{i=1}^{n} \frac{1}{a_i}\left(\frac{a_i}{a_1} - 1\right)\left(1 - \frac{a_i}{a_n}\right),$$

which is true because $a_1 \le a_i \le a_n$ for all $i \in \{1, 2, \dots, n\}$. So the proof is complete.

1210. If $n \ge 1$ then

$$2 + \log\left(\frac{6\big((n+1)! - 1\big)^2}{n(2n^2 + 3n + 1)}\right) \le 2\sqrt{\sum_{k=1}^{n} (k!)^2}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$\sum_{k=1}^{n} (k!)^2 . \sum_{k=1}^{n} k^2 \ge \left(\sum_{k=1}^{n} k.k!\right)^2 = \left(\sum_{k=1}^{n} [(k+1)! - k!]\right)^2 = \big((n+1)! - 1\big)^2.$$

and since $x \ge 1 + \log(x) = \dfrac{2 + \log(x^2)}{2}$, $\forall x > 0$, then

$$2\sqrt{\sum_{k=1}^{n}(k!)^2} \geq 2+\log\left(\sum_{k=1}^{n}(k!)^2\right) \geq 2+\log\left(\frac{6\left((n+1)!-1\right)^2}{n(2n^2+3n+1)}\right).$$

Equality holds if $n=1$.

1211.

If $a, x \in \mathbb{R}$ and $b, b+x > 0$. Find x such that

$$\left(\frac{a}{a+x}\right)^2 - \frac{1}{2}\left(\frac{a}{a+x}-\frac{x}{b}\right)^2 + \left(\frac{b}{b+x}\right)^2 \leq \left(\frac{b}{b+x}+\frac{x}{b}\right)\left(\frac{a}{a+x}-\frac{x}{b}+\frac{1}{2}\right)$$

Proposed by Sidi Abdellah Lemrabott-Mauritania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$0 \geq \left(\frac{a}{a+x}\right)^2 - \frac{1}{2}\left(\frac{a}{a+x}-\frac{x}{b}\right)^2 + \left(\frac{b}{b+x}\right)^2 - \left(\frac{b}{b+x}+\frac{x}{b}\right)\left(\frac{a}{a+x}-\frac{x}{b}+\frac{1}{2}\right)$$

$$= \frac{1}{2}\left[\left(\frac{a}{a+x}\right)^2 - 2\left(\frac{a}{a+x}\right)\left(\frac{b}{b+x}\right) + \left(\frac{b}{b+x}\right)^2\right]$$

$$+\frac{1}{2}\left[\left(\frac{b}{b+x}\right)^2 + \left(\frac{x}{b}\right)^2 + \frac{2x}{b+x} - \frac{b}{b+x} - \frac{x}{b}\right]$$

$$= \frac{1}{2}\left(\frac{a}{a+x}-\frac{b}{b+x}\right)^2 + \frac{1}{2}\left(\frac{b}{b+x}+\frac{x}{b}\right)\left(\frac{b}{b+x}+\frac{x}{b}-1\right)$$

$$= \frac{(a-b)^2x^2}{2(a+x)^2(b+x)^2} + \frac{1}{2}\cdot\frac{b(b+x)+x^2}{b(b+x)}\cdot\frac{x^2}{b(b+x)} \geq 0,$$

$$\Rightarrow \frac{(a-b)^2x^2}{2(a+x)^2(b+x)^2} + \frac{b(b+x)+x^2}{2b^2(b+x)^2}\cdot x^2 = 0.$$

Therefore, $x=0$.

1212. Find all values of $\alpha, \beta \in \mathbb{R}$ such that

$$|a-b| + \alpha(a+b) \geq \sqrt{\beta(a^2+b^2)}, \quad \forall a, b \geq 0$$

Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since the given inequality is true for all $a, b \geq 0$ then it is true for $a > 0, b = 0$ which is

equivalent to $1+\alpha \geq \sqrt{\beta}$, also it is true for $a=b>0$ which is equivalent to
$$\sqrt{2}\alpha \geq \sqrt{\beta}.$$
Now, we will prove that if $\sqrt{\beta} \leq min\{1+\alpha, \sqrt{2}\alpha\}$
then the given inequality is true for all $a, b \geq 0$
The inequality is true if $b=0$. Assume now that $a \geq b > 0$ and let $x := \frac{a}{b} \geq 1$.
The given inequality is equivalent to
$$f(x) = x - 1 + \alpha(x+1) - \sqrt{\beta(x^2+1)}, \quad \forall x \geq 1.$$
We have
$$f'(x) = 1 + \alpha - x\sqrt{\frac{\beta}{x^2+1}} \geq \sqrt{\beta} - \sqrt{\beta\left(1 - \frac{1}{x^2+1}\right)} > 0, \text{so } f \text{ is increasing on } [1,\infty),$$
then $f(x) \geq f(1) = \sqrt{2}(\sqrt{2}\alpha - \sqrt{\beta}) \geq 0, \quad \forall x \geq 1$.
Therefore, the inequality
$|a-b| + \alpha(a+b) \geq \sqrt{\beta(a^2+b^2)}$ is true for all $a, b \geq 0$ if and only if
$$\sqrt{\beta} \leq min\{1+\alpha, \sqrt{2}\alpha\}.$$

1213.

1. Let $a, b > 0$. Prove that
$$|a-b| + \sqrt{a^2 - ab + b^2} \geq \sqrt{2(a^2+b^2) - 3ab}.$$
2. Find value of $\alpha, \beta \in \mathbb{R}$ such that
$$|a-b| + \sqrt{\alpha(a^2 - ab + b^2)} \geq \beta\sqrt{2(a^2+b^2) - 3ab}, \forall a, b \geq 0$$
Proposed by Nguyen Van Canh-BenTre-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

1. We have
$$\sqrt{2(a^2+b^2) - 3ab} - \sqrt{a^2 - ab + b^2} = \frac{(a-b)^2}{\sqrt{2(a^2+b^2) - 3ab} + \sqrt{a^2 - ab + b^2}}$$
$$\leq \frac{(a-b)^2}{0 + \sqrt{a^2 - 2ab + b^2}} = |a-b|,$$
as desired. Equality holds iff $a=b$.

2. Since the given inequality is true for all $a, b \geq 0$ then it is true for $a=0, b>0$ which is
equivalent to $1+\sqrt{\alpha} \geq \sqrt{2}\beta$, also it is true for $a=b>0$ which is equivalent to $\sqrt{\alpha} \geq \beta$.

Now, we will prove that if $\beta \leq min\left\{\frac{1+\sqrt{\alpha}}{\sqrt{2}}, \sqrt{\alpha}\right\}$
then the given inequality is true for all $a, b \geq 0$

• Case 1 : $\sqrt{\alpha} \leq \sqrt{2}+1$. The inequality is true if $a=b=0$.

Assume now that $a \geq b > 0$.

$$\beta\sqrt{2(a^2+b^2)-3ab}-\sqrt{\alpha(a^2-ab+b^2)}$$
$$\leq \sqrt{\alpha[2(a^2+b^2)-3ab]}-\sqrt{\alpha(a^2-ab+b^2)}$$
$$=\frac{\sqrt{\alpha}.(a-b)^2}{\sqrt{2(a^2+b^2)-3ab}+\sqrt{a^2-ab+b^2}}=\frac{\sqrt{\alpha}.(a-b)^2}{\sqrt{2\left(a-\frac{3b}{4}\right)^2+\frac{7b^2}{8}}+\sqrt{\left(a-\frac{b}{2}\right)^2+\frac{3b^2}{4}}}$$
$$\leq \frac{\sqrt{\alpha}.(a-b)^2}{\sqrt{2}\left(a-\frac{3b}{4}\right)+\left(a-\frac{b}{2}\right)}\leq \frac{(\sqrt{2}+1)(a-b)^2}{(\sqrt{2}+1)(a-b)+\frac{(2+\sqrt{2})b}{4}}\leq a-b=|a-b|.$$

• Case 2 : $\sqrt{\alpha}\geq\sqrt{2}+1$. The inequality is true if $a=b=0$.

Assume now that $a \geq b > 0$.

$$\sqrt{2}\left(\beta\sqrt{2(a^2+b^2)-3ab}-\sqrt{\alpha(a^2-ab+b^2)}\right)$$
$$\leq (1+\sqrt{\alpha})\sqrt{2(a^2+b^2)-3ab}-\sqrt{2\alpha(a^2-ab+b^2)}$$
$$=\sqrt{2(a^2+b^2)-3ab}-\sqrt{\alpha}\left(\sqrt{2(a^2+b^2)-2ab}-\sqrt{2(a^2+b^2)-3ab}\right)$$
$$\leq \sqrt{2(a^2+b^2)-3ab}-(\sqrt{2}+1)\left(\sqrt{2(a^2+b^2)-2ab}-\sqrt{2(a^2+b^2)-3ab}\right)$$
$$=\sqrt{2}(\sqrt{2}+1)\left(\sqrt{2(a^2+b^2)-3ab}-\sqrt{a^2-ab+b^2}\right)\leq\sqrt{2}.|a-b|,$$

the proof of the last line is similar to the first case.

So the proof is complete and the condition on α,β is $\beta\leq min\left\{\frac{1+\sqrt{\alpha}}{\sqrt{2}},\sqrt{\alpha}\right\}$

1214. Prove that :

$$\left(\frac{1}{n.e}.\sum_{k=1}^{n}\frac{1}{F_k}\right)^n\geq e^{1-F_{n+2}},\text{ where } F_k \text{ is } k-\text{th Fibonacci number}$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu-Romania

Solution 1 by Tapas Das-India

Let $f(t)=e^t-(t-1), t>0$

$\therefore f'(t)=e^t-1>0$

$f(t)$ is increasing function, $e^t>(t-1)$ (1)

Now

$$\sum_{k=1}^{n}\frac{(1)^2}{F_k}\geq\frac{(1+1+1+\cdots+1)^2}{\sum F_k}=\frac{n^2}{F(n+2)-1}$$

$$\therefore\left(\frac{1}{n\cdot e}\cdot\sum_{k=1}^{n}\frac{1}{F_k}\right)^n\geq\left(\frac{1}{n\cdot e}\cdot\frac{n^2}{F(n+2)-1}\right)^n=\left[\frac{1}{e^{\frac{F(n+2)-1}{n}}}\right]^n$$

$$\geq\left[\frac{1}{e\cdot e^{\frac{F_{n+2}-1}{n}-1}}\right]^n \text{ [using relation (1)]}=\left[e^{\frac{1-F_{n+2}}{n}}\right]^n=e^{(1-F_{n+2})}$$

Note :

$$F(n)=F(n+2)-F(n+1)$$
$$F(n-1)=F(n+1)-F(n)$$
$$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-$$
$$-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-\,-$$
$$F(1)=F(3)-F(2)$$

Sum $=F(n+2)-F(2)$ **---- adding all equation**

$$=F(n+2)-1$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{1}{n.e}\cdot\sum_{k=1}^{n}\frac{1}{F_k}\right)^n\geq e^{1-F_{n+2}}\Leftrightarrow n.\ln\left(\frac{1}{n.e}\cdot\sum_{k=1}^{n}\frac{1}{F_k}\right)\geq 1-F_{n+2}$$

$$\Leftrightarrow \ln\left(\frac{1}{n.e}\cdot\sum_{k=1}^{n}\frac{1}{F_k}\right)\geq\frac{1-F_{n+2}}{n}\Leftrightarrow\frac{1}{n.e}\cdot\sum_{k=1}^{n}\frac{1}{F_k}\overset{(*)}{\geq}e^{\frac{1-F_{n+2}}{n}}$$

$$\text{Now, LHS of }(*)\overset{\text{Bergstrom}}{\geq}\frac{1}{n.e}\cdot\frac{n^2}{\sum_{k=1}^{n}F_k}=\frac{n}{e(F_{n+2}-1)}\overset{?}{\geq}e^{\frac{1-F_{n+2}}{n}}$$

$$\Leftrightarrow\frac{n}{ex}\overset{?}{\geq}e^{\frac{-x}{n}}\ (x=F_{n+2}-1)\Leftrightarrow e^{\frac{x}{n}-1}\underbrace{\overset{?}{\geq}}_{(**)}\frac{x}{n}$$

$$\text{But, }\because e^{\theta}\geq 1+\theta\ \forall\,\theta\in\mathbb{R}\ \therefore e^{\frac{x}{n}-1}\geq 1+\frac{x}{n}-1=\frac{x}{n}\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\frac{1}{n.e}\cdot\sum_{k=1}^{n}\frac{1}{F_k}\right)^n\geq e^{1-F_{n+2}}\ \text{(QED)}$$

1215. **If $x>-1$ then:**

$$\frac{x^2}{2}\cdot\min\left(1,\frac{1}{1+x}\right)\leq x-\ln(1+x)\leq\frac{x^2}{2}\cdot\max\left(1,\frac{1}{1+x}\right)$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Ravi Prakash-New Delhi-India

For $x>-1$, let

$$f(x)=x-\log(1+x)-\frac{x^2}{2}$$

$$f'(x)=1-\frac{1}{1+x}-x=\frac{1-x^2-1}{1+x}=\frac{-x^2}{1+x}$$

$$\Rightarrow f'(x)<0;\forall x>-1,x\neq 0\Rightarrow f(x)\text{ decreases on }(-1,\infty)$$

For $-1 < x \le 0, f(x) \le f(x) \Rightarrow 0 \le x - \log(1+x) - \frac{x^2}{2}$

$$\Rightarrow \frac{x^2}{2} \le x - \log(1+x)$$

For $x > 0, f(x) < f(0) \Rightarrow x - \log(1+x) - \frac{x^2}{2} < 0 \Rightarrow x - \log(1+x) < \frac{x^2}{2}$

For $x > -1$, let

$$g(x) = x - \log(1+x) - \frac{x^2}{2(1+x)} = x - \log(1+x) - \frac{x^2 - 1 + 1}{2(1+x)}$$

$$= x - \log(1+x) - \frac{1}{2}(x-1) - \frac{1}{2(1+x)} = \frac{1}{2}(x+1) - \log(1+x) - \frac{1}{2(1+x)}$$

$$g'(x) = \frac{1}{2} - \frac{1}{1+x} + \frac{1}{2(1+x)^2} = \frac{(1+x)^2 - 2(1+x) + 1}{2(1+x)^2}$$

$= \frac{x^2}{2(1+x)^2} > 0, \forall x > 01, x \ne 0 \Rightarrow g(x)$ increases on $(-1, \infty)$

For $-1 < x \le 0, g(x) \le g(0) = 0 \Rightarrow x - \log(1+x) \le \frac{x^2}{2(1+x)}$

For $x > 0, g(x) > g(0) \Rightarrow x - \log(1+x) - \frac{x^2}{2(1+x)} > 0$

$$\Rightarrow x - \log(1+x) > \frac{x^2}{2(1+x)}$$

Thus, for $-1 < x \le 0$

$$\frac{x^2}{2} \le x - \log(1+x) < \frac{x^2}{2(1+x)}$$

For $-1 < x \le 0$, $\min\left\{1, \frac{1}{1+x}\right\} = 1$ and $\max\left\{1, \frac{1}{1+x}\right\} = \frac{1}{1+x}$

Thus, for $-1 < x \le 0$

$$\frac{x^2}{2} \min\left\{1, \frac{1}{1+x}\right\} \le x - \log(1+x) \le \frac{x^2}{2} \max\left\{1, \frac{1}{1+x}\right\} \quad (1)$$

For $x > 0$,

$$\frac{x^2}{2(1+x)} \le x - \log(1+x) \le \frac{x^2}{2}$$

Also, for $x > 0$

$$\min\left\{1, \frac{1}{1+x}\right\} = \frac{1}{1+x} \text{ and } \max\left\{1, \frac{1}{1+x}\right\} = 1$$

Thus, for $x > 0$,

$$\frac{x^2}{2}\min\left\{1,\frac{1}{1+x}\right\} \le x - \log(1+x) \le \frac{x^2}{2}\max\left\{1,\frac{1}{1+x}\right\} \quad (2)$$

From (1) and (2), inequality follows for all $x > -1$.

1216. If $x, y, z > 0$ and $p, q, r \in \left[0, \frac{1}{2}\right]$ are such that

$$x + y + z = p + q + r = 1 \text{ then prove that: } xyz \le \frac{1}{8}(px + qy + rz)$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by proposer

Let's pay attention: $\frac{1}{2} - q \le p \le \frac{1}{2}$

Then for the function: $f(p) = px + qy + (1 - p - q)z$

We have: $f(p) \ge \min\left[f\left(\frac{1}{2} - q\right), f\left(\frac{1}{2}\right)\right]$

$$= \min\left[\left(\frac{1}{2} - q\right)x + qy + \frac{1}{2}z, \frac{1}{2}x + qy + \left(\frac{1}{2} - q\right)z\right]$$

$$\ge \min\left(\frac{1}{2}x + \frac{1}{2}z, \frac{1}{2}y + \frac{1}{2}z, \frac{1}{2}x + \frac{1}{2}y\right)$$

Let: $\min\left(\frac{1}{2}x + \frac{1}{2}z, \frac{1}{2}y + \frac{1}{2}z, \frac{1}{2}x + \frac{1}{2}y\right) = \frac{1}{2}(x + z)$

Consequently: $px + qy + rz \ge \frac{1}{2}(x + z)$

$$\ge (x + z)2(x + z)\big(1 - (x + z)\big) = 2(x + z)^2 y \ge 8xyz$$

or $\Rightarrow xyz \le \frac{1}{8}(px + qy + rz)$

1217. If $n \in \{2, 3, \dots\}$ and $x \in (0, 1)$, then prove that

$$\frac{8\left(\frac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n}\right)^2}{2x + n + 1} + \ln\left(\frac{x}{\sqrt[n]{n!}}\right) < 2$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Firstly, we will prove the lemma that if $x \in (0, 1)$ then we have $\frac{4}{x + 1} + \ln(x) < 2$.

Let $f(t) = \frac{4}{t + 1} + \ln(t), t \in (0, 1]$.

We have $f'(t) = -\frac{4}{(t + 1)^2} + \frac{1}{t} = \frac{(t - 1)^2}{t(t + 1)^2} \ge 0$, then f is

strictly increasing on $(0, 1]$, thus $\frac{4}{x + 1} + \ln(x) = f(x) < f(1) = 2, \quad \forall x \in (0, 1)$.

Now let $k \in \{1,2,\dots,n\}$. Since $\frac{x}{k} \in (0,1)$, then $\frac{4}{\frac{x}{k}+1} + ln\left(\frac{x}{k}\right) < 2,\ \ \forall k \in \{1,2,\dots,n\}$.

Adding this result for $k = 1,2,\dots,n$, we obtain

$$4\sum_{k=1}^{n}\frac{k}{x+k} + ln\left(\frac{x^n}{n!}\right) < 2n \text{ or } \frac{4}{n}\sum_{k=1}^{n}\frac{k}{x+k} + ln\left(\frac{x}{\sqrt[n]{n!}}\right) < 2.$$

By CBS inequality, we have

$$\sum_{k=1}^{n}\frac{k}{x+k} \geq \frac{\left(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}\right)^2}{\sum_{k=1}^{n}(x+k)} = \frac{n\left(\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n}\right)^2}{x+\frac{n+1}{2}}.$$

Therefore

$$\frac{8\left(\frac{\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}}{n}\right)^2}{2x+n+1} + ln\left(\frac{x}{\sqrt[n]{n!}}\right) < 2.$$

1218. If $0 \leq a,b,c,d \leq 4, a^2+b^2+c^2+d^2 = 30$ then find $min\Omega$

$$\Omega = \sqrt{a^3+b^3+c^3} + \sqrt{b^3+c^3+d^3} + \sqrt{c^3+d^3+a^3} + \sqrt{d^3+a^3+b^3}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $s := a+b+c+d$.

By CBS inequality, we have

$$\sqrt{b^3+c^3+d^3} = \frac{\sqrt{(b^3+c^3+d^3)(b+c+d)}}{\sqrt{b+c+d}} \geq \frac{b^2+c^2+d^2}{\sqrt{b+c+d}} = \frac{30-a^2}{\sqrt{s-a}}.$$

then by using Jensen's inequality on the convex function $x \to \frac{1}{\sqrt{x}}$, we have

$$\Omega \geq \sum_{cyc}(30-a^2).\frac{1}{\sqrt{s-a}} \geq \sqrt{\frac{\left(\sum_{cyc}(30-a^2)\right)^3}{\sum_{cyc}(30-a^2)(s-a)}} = \sqrt{\frac{90^3}{60s+\sum_{cyc}a^3}}$$

$$= \frac{270\sqrt{10}}{\sqrt{\sum_{cyc}(a^3+60a)}},$$

and since

$$a^3+60a = \frac{11\sqrt{30}}{4}a^2 + \frac{105\sqrt{30}}{8} - \left(\frac{7\sqrt{30}}{4} - a\right)\left(a - \frac{\sqrt{30}}{2}\right)^2 \leq \frac{\sqrt{30}}{8}(22a^2+105),$$

then

$$\Omega \geq \frac{270\sqrt{10}}{\sqrt{\sum_{cyc}\frac{\sqrt{30}}{8}(22a^2+105)}} = \frac{270\sqrt{10}}{\sqrt{135\sqrt{30}}} = 6\sqrt[4]{750}.$$

Therefore $min\Omega = 6\sqrt[4]{750}$ reached at $a = b = c = d = \frac{\sqrt{30}}{2}$.

1219.

Let $a, b, c \geq 0$ such that $a + b + c = 1$. Find the maximum value of P :

$$P = \sqrt{3a^2 + a + 1} + \sqrt{3b^2 + b + 1} + \sqrt{3c^2 + c + 1}$$

Proposed by Tran Quoc Thinh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $f(x) = \sqrt{3x^2 + x + 1}$, $x \in [0, 1]$.

We have $f''(x) = \frac{11}{4\sqrt{(3x^2 + x + 1)^3}} > 0$, then f is convex on $[0, 1]$, which implies that

$$\sqrt{3x^2 + x + 1} = f(x) = f\big((1 - x).0 + x.1\big) \leq (1 - x).f(0) + x.f(1)$$
$$= 1 + (\sqrt{5} - 1)x, x \in [0, 1]$$

Similarly, we have

$$\sqrt{3b^2 + b + 1} \leq 1 + (\sqrt{5} - 1)b \text{ and } \sqrt{3c^2 + c + 1} \leq 1 + (\sqrt{5} - 1)c.$$

then

$$P \leq 3 + (\sqrt{5} - 1)(a + b + c) = 2 + \sqrt{5}.$$

So the maximum value of P is $2 + \sqrt{5}$ reached at $a = 1, b = c = 0$ and permutations.

1220. If $x \geq 0, a_1, a_2, \ldots, a_n > 0$ then:

$$\sum_{i=1}^{n}\left(a_i^{\cosh x} \cdot b_i^{\sinh x}\right)^{\frac{1}{e^x}} \leq \left(\left(\sum_{i=1}^{n} a_i\right)^{\cosh x} \cdot \left(\sum_{i=1}^{n} b_i\right)^{\sinh x}\right)^{\frac{1}{e^x}}$$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Adrian Popa

$$\left(\sum_{i=1}^{n} a_i\right)^{\frac{\cosh x}{e^x}} \cdot \left(\sum_{i}^{n} b_i\right)^{\frac{\sinh x}{e^x}} = \left(\sum_{i=1}^{n}\left(a_i^{\frac{\cosh x}{e^x}}\right)^{\frac{e^x}{\cosh x}}\right)^{\frac{\cosh x}{e^x}} \cdot \left(\sum_{i}^{n}\left(b_i^{\frac{\sinh x}{e^x}}\right)^{\frac{e^x}{\sinh x}}\right)^{\frac{\sinh x}{e^x}} \geq$$

$$\therefore \left[\begin{matrix} \cosh x = \dfrac{e^x + e^{-x}}{2} \\ \sinh x = \dfrac{e^x - e^{-x}}{2} \end{matrix}\right\} \Rightarrow \sinh x + \cosh x = e^x \Rightarrow \frac{\sinh x}{e^x} + \frac{\cosh x}{e^x} = 1$$

$$\overset{J.Holder}{\geq} \sum_{i=1}^{n} a_i^{\frac{\cosh x}{e^x}} \cdot b_i^{\frac{\sinh x}{e^x}} = \sum_{i=1}^{n} \left(a_i^{\sinh x} \cdot b_i^{\cosh x}\right)^{\frac{1}{e^x}}$$

Holder's inequality: $\sum_i^n x_i y_i \leq \left(\sum_i^n x_i^p\right)^{\frac{1}{p}} \cdot \left(\sum_i^n y_i^p\right)^{\frac{1}{p}}$ with $\frac{1}{p} + \frac{1}{q} = 1$

In our case: $\left.\begin{matrix} x_i = a_i^{\frac{\cosh x}{e^x}} & p = \dfrac{e^x}{\cosh x} \\ y_i = b_i^{\frac{\sinh x}{e^x}} & q = \dfrac{e^x}{\sinh x} \end{matrix}\right\} \Rightarrow \frac{1}{p} + \frac{1}{q} = 1$

1221. If $x, y > 0$ then: $\Gamma(x+5) \cdot \Gamma(y+7) \leq 11! \cdot \Gamma(x+y)$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

If $x; y > 0$ then $\Gamma(x+5) \cdot \Gamma(y+7) \leq 11! \cdot \Gamma(x+y)$

The inequality holds if: $0 < x < 7, 0 < y < 5$ or $x > 7, y > 5$

If $x = 7$ or $y = 5$ the inequality is satisfied with equality

If $0 < x < 7, y > 5$ or $x > 7, 0 < y < 5$ the inequality is reversed

Consider the functions: $\alpha(u), \mu(u), c(u), u \in [0; 1], \alpha(u) \geq 0,$

$\mu(u), c(u)$ have opposite monotonicity

If μ is increasing and c is decreasing : $\begin{matrix} u > v \text{ then } \mu(u) - \mu(v) \geq 0, c(u) - c(v) \leq 0 \\ u < v \text{ then } \mu(u) - \mu(v) \leq 0, c(u) - c(v) \geq 0 \end{matrix}$

If μ is decreasing and c is increasing: $\begin{matrix} u > v \text{ then } \mu(u) - \mu(v) \leq 0, c(u) - c(v) \geq 0 \\ u < v \text{ then } \mu(u) - \mu(v) \geq 0, c(u) - c(v) \leq 0 \end{matrix}$

$\forall u, v \in [0; 1]$ and $\big(\mu(u) - \mu(v)\big)\big(c(u) - c(v)\big) \leq 0$

Since $\alpha(u) \geq 0$ then we have

$$\alpha(u)\alpha(v)\big(\mu(u) - \mu(v)\big)\big(c(u) - c(v)\big) \leq 0, \forall u, v \in [0; 1]$$

Integrate, $\int_0^1 \int_0^1 du\, dv \Rightarrow$

$$\int_0^1 \alpha(v)\, dv \int_0^1 \alpha(u)\mu(u)\, c(u) du - \int_0^1 \alpha(u)\, \mu(u) du \int_0^1 \alpha(v) c(v)\, dv -$$

$$-\int_0^1 \alpha(u)c(u)\,du\int_0^1 \alpha(v)\,\mu(v)dv+\int_0^1 \alpha(u)\,du\int_0^1 \alpha(v)\,\mu(v)c(v)dv\le 0$$

$$\Rightarrow \int_0^1 \alpha(u)\,du\int_0^1 \alpha(u)\mu(u)c(u)\,du\le \int_0^1 \alpha(u)\,\mu(u)du\int_0^1 \alpha(u)c(u)\,du$$

(i) Set $\underbrace{\alpha(u)=u^{x-1}(1-u)^{y-1}}_{\ge 0},\underbrace{\mu(u)=u^{7-x}}_{increasing},\underbrace{c(u)=(1-u)^{5-y}}_{decreasing},0<x<7,0<y<5$

$\mu(u),c(u)$ have oppositive monotonicity

$$\int_0^1 u^{x-1}\,(1-u)^{y-1}du\int_0^1 u^{x-1}\,(1-u)^{y-1}u^{7-x}(1-u)^{5-y}du=\beta(x,y)\beta(7,5)=$$

$$=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$\int_0^1 u^{x-1}\,(1-u)^{y-1}u^{7-x}du\int_0^1 u^{x-1}\,(1-u)^{y-1}(1-u)^{5-y}du=\beta(7,y)\beta(x,5)=$$

$$=\frac{\Gamma(7)\Gamma(\mathrm{y})}{\Gamma(7+\mathrm{y})}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(5+\mathrm{x})}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}\le\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\Rightarrow \Gamma(x+y)\Gamma(12)\ge\Gamma(7+y)\Gamma(5+x)$$

(ii) Set $\underbrace{\alpha(u)=u^{7-1}(1-u)^{5-1}}_{\ge 0},\underbrace{\mu(u)=u^{x-7}}_{increasing},\underbrace{c(u)=(1-u)^{y-5}}_{decreasing},x>7,y>5$

$\mu(u),c(u)$ have oppositive monotonicity

$$\int_0^1 u^{7-1}\,(1-u)^{5-1}du\int_0^1 u^{7-1}\,(1-u)^{5-1}u^{x-7}(1-u)^{y-5}du=\beta(7,5)\beta(x,y)=$$

$$=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$\int_0^1 u^{7-1}(1-u)^{5-1}\,u^{x-7}du\int_0^1 u^{7-1}\,(1-u)(1-u)^{y-5}du=\beta(x,5)\beta(7,y)=$$

$$=\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+u)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\cdot\frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}\le\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+u)}\cdot\frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)}$$

$$\Rightarrow \Gamma(x+y)\Gamma(12)\ge\Gamma(7+y)\Gamma(5+x)$$

(iii) Set $\alpha(u) = u^{x-1}\underset{\geq 0}{(1-u)^{5-1}}, \mu(u) = \underset{increasing}{u^{7-x}}, c(u) = \underset{decreasing}{(1-u)^{y-5}}, 0 < x < 7, y > 5$

$\mu(u), c(u)$ **have oppositive monotonicity**

$$\int_0^1 u^{x-1}(1-u)^{5-1}\,du \int_0^1 u^{x-1}(1-u)^{5-1}u^{7-x}(1-u)^{y-5}du = \beta(x,5)\beta(7,y) =$$

$$= \frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)} \cdot \frac{\Gamma(y)\Gamma(7)}{\Gamma(7+y)}$$

$$\int_0^1 u^{x-1}(1-u)^{5-1}u^{7-x}du \int_0^1 u^{x-1}(1-u)^{5-1}(1-u)^{y-5}du = \beta(7,5)\beta(x,y) =$$

$$= \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)} \cdot \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)} \cdot \frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)} \leq \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot \frac{\Gamma(7)\Gamma(5)}{\Gamma(12)}$$

$$\Rightarrow \Gamma(x+y)\Gamma(12) \leq \Gamma(7+y)\Gamma(5+x)$$

(iv) Set $\alpha(u) = u^{7-1}\underset{\geq 0}{(1-u)^{y-1}}, \mu(u) = \underset{increasing}{u^{x-7}}, c(u) = \underset{decreasing}{(1-u)^{5-y}}, x > 7, 0 < y < 5$

$\mu(u), c(u)$ **have oppositive monotonicity**

$$\int_0^1 u^{7-1}(1-u)^{y-1}du \int_0^1 u^{7-1}(1-u)^{y-1}u^{x-7}(1-u)^{5-y}du = \beta(7,y)\beta(x,5) =$$

$$= \frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)} \cdot \frac{\Gamma(5)\Gamma(x)}{\Gamma(5+x)}$$

$$\int_0^1 y^{7-1}(1-u)^{y-1}u^{x-7}du \int_0^1 u^{7-1}(1-u)^{y-1}(1-u)^{5-y}du = \beta(x,y)\beta(7,5) =$$

$$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x,y)} \cdot \frac{\Gamma(7)\Gamma(5)}{\Gamma(7+5)}$$

$$\frac{\Gamma(7)\Gamma(y)}{\Gamma(7+y)} \cdot \frac{\Gamma(x)\Gamma(5)}{\Gamma(5+x)} \leq \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \cdot \frac{\Gamma(7+5)}{\Gamma(12)}$$

$$\Rightarrow \Gamma(x+y)\Gamma(12) \leq \Gamma(7+y)\Gamma(5+x)$$

Therefore $\Rightarrow \Gamma(x+5)\cdot\Gamma(y+7) \leq 11!\cdot\Gamma(x+y)$

1222.If $a,b,c \geq 0$ **such that** $a+b+c+\sqrt[3]{abc} = 2$.

Find the maximum value of $P = \sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\sqrt[3]{abc}$

Proposed by Phan Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Applying Schur's inequality on the triple $(\sqrt[3]{a},\sqrt[3]{b},\sqrt[3]{c})$, we have

$$a+b+c+3\sqrt[3]{abc}\ge\sqrt[3]{ab}(\sqrt[3]{a}+\sqrt[3]{b})+\sqrt[3]{bc}(\sqrt[3]{b}+\sqrt[3]{c})+\sqrt[3]{ca}(\sqrt[3]{c}+\sqrt[3]{a})$$

$$\overset{AM-GM}{\ge}\sqrt[3]{ab}.2\sqrt[6]{ab}+\sqrt[3]{bc}.2\sqrt[6]{bc}+\sqrt[3]{ca}.2\sqrt[6]{ca}=2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}).$$

Then

$$P=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\sqrt[3]{abc}\le\frac{a+b+c+\sqrt[3]{abc}}{2}=1.$$

The maximum value of P is 1, it reaches at

$$a=b=c=\frac{1}{2}\text{ or }a=b=1,c=0\text{ and permutations.}$$

1223. Let $a,b,c\ge 0$ such that $a+b+c+\sqrt[3]{abc}=4$.

Find the maximum value of P:

$$P=\sqrt{a(4a+3\sqrt[3]{abc}+2)}+\sqrt{b(4b+3\sqrt[3]{abc}+2)}+\sqrt{c(4c+3\sqrt[3]{abc}+2)}$$

Proposed by Pham Ngoc Chau-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

If two of the numbers a,b,c are zero then we have $P=6\sqrt{2}$. Assume now that $ab+bc+ca>0$.

By AM − GM inequality, we have

$$2\sqrt{a(4a+3\sqrt[3]{abc}+2)}\le\sqrt[3]{a^2}(2\sqrt[3]{a}+\sqrt[6]{bc})+\frac{\sqrt[3]{a}.(4a+3\sqrt[3]{abc}+2)}{2\sqrt[3]{a}+\sqrt[6]{bc}}$$

$$=4a+\frac{2\sqrt[3]{a}.(2\sqrt[3]{abc}+1)}{2\sqrt[3]{a}+\sqrt[6]{bc}}$$

$$\le 4a+\frac{2\sqrt[3]{a}.(2\sqrt[3]{abc}+1)}{2\sqrt[3]{a}+\frac{2\sqrt[3]{bc}}{\sqrt[3]{b}+\sqrt[3]{c}}}=4a+\frac{(\sqrt[3]{ab}+\sqrt[3]{ca})(2\sqrt[3]{abc}+1)}{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}.$$

Then

$$\sqrt{a(4a+3\sqrt[3]{abc}+2)}\le 2a+\frac{\sqrt[3]{ab}+\sqrt[3]{ca}}{2(\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca})}.(2\sqrt[3]{abc}+1)\ \text{(and analogs)}$$

Adding this inequality with the similar ones, we obtain

$$P\le 2a+2b+2c+2\sqrt[3]{abc}+1=9.$$

So the maximum value of P is 9, it reaches at $a=b=c=1$.

1224. If $0\le x,y,z\le\frac{\pi}{4}, x+y+z=\frac{\pi}{4}$ then:

$$1+\tan x\cdot\tan y\cdot\tan z>4\sqrt{\tan x\cdot\tan y\cdot\tan z}$$

Proposed by Daniel Sitaru – Romania

Solution by Tapas Das – India

$$x+y+z=\frac{\pi}{4}$$

$$\therefore \tan(x+y+z)=1$$

$$\frac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan x\tan y-\tan y\tan z-\tan z\tan x}=1$$

$$\therefore 1+\tan x\tan y\tan z$$

$$=\tan x+\tan y+\tan z+\tan x\tan y+\tan y\tan z+\tan z\tan x$$

$$\overset{AM-GM}{\geq} 3(\tan x\tan y\tan z)^{\frac{1}{3}}+3(\tan x\tan y\tan z)^{\frac{2}{3}}$$

$$\overset{AM-GM}{\geq} 6\cdot\sqrt{(\tan x\tan y\tan z)^{\frac{1}{3}}\cdot(\tan x\tan y\tan z)^{\frac{2}{3}}}=6\sqrt{\tan x\tan y\tan z}$$

$$>4\sqrt{\tan x\tan y\tan z}$$

1225. If $1\leq a,b,c\leq 2, 3\leq d,e,f\leq 4$ then:

$$(a^2+b^2+c^2)(d^2+e^2+f^2)\leq(ad+be+cf)^2+81$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Sanong Huayrerai-Nakon Pathom-Thailand

Iff $a^2e^2+a^2f^2+b^2d^2+b^2f^2+c^2d^2+c^2e^2$

$\leq 2(abde+acdf+bcef)+81$ ok

Because $a^2e^2+b^2d^2\leq 2abde+27$

$a^2f^2+c^2d^2\leq 2acdf+27$

and $b^2f^2+c^2e^2\leq 2bcef+27$

Because $a^2e^2+b^2d^2\leq 2cbde+27$

Iff $a^2e^2abde+b^2d^2-abde\leq 27$

Iff $ae\,(ae-bd)+bdcbf-ae\leq 27$

$(ae-bd)^2\leq 27$ ok

Because $1\leq ab\leq 2, 3\leq de\leq 4$

Hence $3\leq ae\leq 8, 8\leq bd\leq 8, -8\leq -bd\leq -3$

$-f\leq ae-bd\leq 5, (ae-bd)^2\leq 25\leq 27$

Therefore it is to be true

Solution 2 by Tapas Das-India

$$(a^2+b^2+c^2)(d^2+e^2+f^2)-(ad+be+cf)^2-81$$

$$=(ac-bd)^2+(bf-ec)^2+(af-cd)^2-81<25+25+25-81<0$$

$$\therefore (a^2+b^2+c^2)(d^2+c^2+f^2)\le(ad+be+cf)^2+81$$

Now $1<a<2,\ 3<e<4,\ 3<ae<8, 1<b<2, 3<d<4$

$$3<bd<8,\qquad -8<-bd<-3$$

$$\therefore (ae-bd)<8-3=5$$

$$\therefore (ae-bd)^2<25$$

$$\therefore (bf-ec)^2<25$$

$$(af-cd)^2<25$$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

Suppose: $\vec{u}(a,b,c),\vec{v}(d,e,f)$:

$\|\vec{u}\|^2=a^2+b^2+c^2, \|\vec{v}\|^2=d^2+e^2+f^2$ so:

$$\vec{u}\cdot\vec{v}=ad+be+cf$$

$$\|\vec{u}\|\cdot\|\vec{v}\|^2\overset{?}{\le}(\vec{u}\cdot\vec{v})^2+81$$

$$\|\vec{u}\|^2\cdot\|\vec{v}\|^2\overset{?}{\le}\|\vec{u}\|^2\cdot\|\vec{v}\|^2\cdot\cos^2\theta+81$$

$$\|\vec{u}\|\cdot\|\vec{v}\|(1-\cos^2\theta)\overset{?}{\le}81$$

$$\|\vec{u}\|^2\cdot\|\vec{v}\|^2\cdot\sin^2\theta\overset{?}{\le}81\Rightarrow\|\vec{u}\wedge\vec{v}\|^2\overset{?}{\le}81$$

$$\vec{u}\wedge\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a & b & c\\ d & e & f\end{vmatrix}=(bf-ce)\vec{i}-(af-dc)\vec{j}+(ae-bd)\vec{k}$$

$$\|\vec{v}\wedge\vec{v}\|^2=(bf-ce)^2+(af-dc)^2-(ae-bd)^2$$

Because: $1\le a,b,c\le 2, 3\le d,e,f\le 4$

$$3\le a\cdot e\le 8, 3\le bd\le 8\Rightarrow -8\le -bd\le -3\Rightarrow -s\le ae-bd\le 5$$

$(ae-bd)^2\le 25\le 27$ in a similar way $(ab-dc)^2\le 27$

$$(ae-bd)^2\le 27$$

So: $\|\vec{u}\wedge\vec{v}\|\le 3\cdot 27$

$$\|\vec{u}\wedge v\|^2\le 81$$

1226. If $a,b,c>0$ such that $abc=1$, then prove that :

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\ge\sqrt[3]{24+\sqrt[3]{27+\sqrt{\det\begin{pmatrix}\frac{2}{a}-a^2 & c^2 & b^2\\ c^2 & \frac{2}{b}-b^2 & a^2\\ b^2 & a^2 & \frac{2}{c}-c^2\end{pmatrix}}}}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\det\begin{pmatrix}\frac{2}{a}-a^2 & c^2 & b^2\\ c^2 & \frac{2}{b}-b^2 & a^2\\ b^2 & a^2 & \frac{2}{c}-c^2\end{pmatrix}\overset{abc=1}{=}$$

$$(2bc-a^2)\left((2ca-b^2)(2ab-c^2)-a^4\right)+c^2\left(a^2b^2-c^2(2ab-c^2)\right)$$
$$+b^2\left(c^2a^2-b^2(2ca-b^2)\right)$$
$$=a^6+b^6+c^6+2a^3b^3+2b^3c^3+2c^3a^3+9a^2b^2c^2-6abc(a^3+b^3+c^3)$$
$$=\left(\sum_{cyc}a^3\right)^2+9a^2b^2c^2-6abc\left(\sum_{cyc}a^3\right)=\left(\sum_{cyc}a^3-3abc\right)^2$$

$$\therefore\sqrt[3]{24+\sqrt[3]{27+\sqrt{\det\begin{pmatrix}\frac{2}{a}-a^2 & c^2 & b^2\\ c^2 & \frac{2}{b}-b^2 & a^2\\ b^2 & a^2 & \frac{2}{c}-c^2\end{pmatrix}}}}=$$

$$\sqrt[3]{24+\sqrt[3]{27+\sqrt{\left(\sum_{cyc}a^3-3abc\right)^2}}}\overset{abc=1}{=}\sqrt[3]{24+\sqrt[3]{\sum_{cyc}a^3+24abc}}$$

$$\overset{?}{\le}\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\Leftrightarrow\sum_{cyc}x\overset{?}{\ge}\sqrt[3]{24xyz+\sqrt[3]{\sum_{cyc}x^9+24x^3y^3z^3}}$$

$$\left(x=\sqrt[3]{a},y=\sqrt[3]{b},z=\sqrt[3]{c}\text{ and }xyz=1\because abc=1\right)$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^3-24xyz\overset{?}{\geq}\sqrt[3]{\sum_{cyc}x^9+24x^3y^3z^3}$$

$$\Leftrightarrow\left(\left(\sum_{cyc}x\right)^3-24xyz\right)^3\overset{?}{\underset{(\bullet)}{\geq}}\sum_{cyc}x^9+24x^3y^3z^3$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ and $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y$
$\Rightarrow$ X, Y, Z form sides of a triangle with semiperimeter, circumradius and inradius $=s, R, r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}X=2s\Rightarrow\sum_{cyc}x=s\to(1)$
$\Rightarrow x=s-X, y=s-Y, z=s-Z$ and such substitutions $\Rightarrow$
$\sum_{cyc}xy=\sum_{cyc}(s-X)(s-Y)\Rightarrow\sum_{cyc}xy=4Rr+r^2\to(2)$ and

$$\sum_{cyc}x^3=\left(\sum_{cyc}x\right)^3-3\prod_{cyc}(y+z)\overset{\text{via }(1)}{=}s^3-3XYZ=s^3-12Rrs$$

$$\Rightarrow\sum_{cyc}x^3=s^3-12Rrs\to(3)\text{ and also},\sum_{cyc}x^3y^3=\left(\sum_{cyc}xy\right)^3-3xyz\prod_{cyc}(y+z)$$

$$\overset{\text{via }(2)\text{ and }(3)}{=}(4Rr+r^2)^3-3r^2s.4Rrs\Rightarrow\sum_{cyc}x^3y^3=(4Rr+r^2)^3-12Rr^3s^2\to(4)$$

$$\therefore\sum_{cyc}x^9=\left(\sum_{cyc}x^3\right)^3-3(x^3+y^3)(y^3+z^3)(z^3+x^3)$$

$$=\left(\sum_{cyc}x^3\right)^3-3\left(\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x^3y^3\right)-x^3y^3z^3\right)$$

$$\therefore(*)\Leftrightarrow\left(\left(\sum_{cyc}x\right)^3-24xyz\right)^3\overset{?}{\geq}$$

$$\left(\sum_{cyc}x^3\right)^3-3\left(\left(\sum_{cyc}x^3\right)\left(\sum_{cyc}x^3y^3\right)-x^3y^3z^3\right)+24x^3y^3z^3$$

$$\overset{\text{via }(1),(3)\text{ and }(4)}{\Leftrightarrow}(s^3-24r^2s)^3\overset{?}{\geq}(s^3-12Rrs)^3+27r^6s^3$$

$$-3(s^3-12Rrs)\left((4Rr+r^2)^3-12Rr^3s^2\right)$$

$$\Leftrightarrow(3R-6r)s^6-r(36R^2+3Rr-144r^2)s^4$$

$$+r^2s^2(160R^3+48R^2r+3Rr^2-1154r^3)-3Rr^3(4R+r)^3\overset{?}{\underset{(*)}{\geq}}0\text{ and}$$

$$\because (3R-6r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen + Euler}}{\geq} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \geq (3R-6r)(s^2-16Rr+5r^2)^3$$

$$\Leftrightarrow (108R-120r)(R-2r)s^4 - 6r^2s^4 + rs^2(160R^3+48R^2r+3Rr^2-1154r^3)$$

$$-3Rr^2(4R+r)^3 \underbrace{\overset{?}{\geq}}_{(**)} 0$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (108R-120r)(R-2r)(16Rr-5r^2)s^2$$

$$-6r^2s^2(4R^2+4Rr+3r^2) + rs^2(160R^3+48R^2r+3Rr^2-1154r^3)$$

$$-3Rr^2(4R+r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow (1888R^3-5892R^2r+5499Rr^2-2372r^3)s^2 \underbrace{\overset{?}{\geq}}_{(***)} 3Rr(4R+r)^3$$

$$\because 1888R^3-5892R^2r+5499Rr^2-2372r^3$$

$$= (R-2r)(1888R^2-2116Rr+1267r^2)+162r^3 \overset{\text{Euler}}{\geq} 162r^3 > 0$$

$$\therefore \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq}$$

$$(1888R^3-5892R^2r+5499Rr^2-2372r^3)(16Rr-5r^2) \overset{?}{\geq} 3Rr(4R+r)^3$$

$$\Leftrightarrow 30016t^4-103856t^3+117408t^2-65450t+11860 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(30016t^2+16208t+62176)+118422\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \geq \sqrt[3]{24+\sqrt[3]{27+\sqrt{\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix}}}}$$

$$\forall a,b,c>0 \mid abc=1, ''='' \text{ iff } a=b=c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have

$$\det\begin{pmatrix} \frac{2}{a}-a^2 & c^2 & b^2 \\ c^2 & \frac{2}{b}-b^2 & a^2 \\ b^2 & a^2 & \frac{2}{c}-c^2 \end{pmatrix} = \det\begin{pmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{pmatrix}$$

$$= (2bc-a^2)[(2ca-b^2)(2ab-c^2)-a^4] - c^2[c^2(2ab-c^2)-a^2b^2]$$

$$+ b^2[c^2a^2-b^2(2ca-b^2)]$$

$$= a^6+b^6+c^6+2(a^3b^3+b^3c^3+c^3a^3)-6abc(a^3+b^3+c^3)+9(abc)^2$$

$$= (a^3+b^3+c^3-3abc)^2 = (a^3+b^3+c^3-3)^2.$$

So it suffices to prove that

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}\geq\sqrt[3]{24+\sqrt[3]{24+a^3+b^3+c^3}}.$$

By AM − GM inequality, we have

$$\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=\sqrt[3]{a+b+c+3\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{b}+\sqrt[3]{c}\right)\left(\sqrt[3]{c}+\sqrt[3]{a}\right)}$$

$$\geq\sqrt[3]{a+b+c+3.8\sqrt[3]{abc}}=\sqrt[3]{\sqrt[3]{a^3+b^3+c^3+3(a+b)(b+c)(c+a)}+24}$$

$$\geq\sqrt[3]{\sqrt[3]{a^3+b^3+c^3+3.8abc}+24}=\sqrt[3]{24+\sqrt[3]{24+a^3+b^3+c^3}},$$

so the proof is complete. Equality holds iff $a=b=c=1$.

1227. If $a,b\in\mathbb{C},n\in\mathbb{N}^*$ then:

$$|a+b-2-2i|^n\leq 2^{n-1}(|a-1-i|^n+|b-1-i|^n)$$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$|a+b-2-2i|^n\leq 2^{n-1}(|a-1-i|^n+|b-1-i|^n)$$

We will use Clarkson's Inequality

If we let

$z=a-(1+i)$ and $w=b-(1+i)$

Then

$$|a+b-2-2i|^n=|z+w|^n\leq|z+w|^n+|z-w|^n\leq$$

$$\leq 2^{n-1}(|z|^n+|w|^n)=2^{n-1}(|a-1-i|^n+|b-1-i|^n)$$

Hence

$$|a+b-2-2i|^n\leq 2^{n-1}(|a-1-i|^n+|b-1-i|^n)$$

Note

The second-to-last inequality follows Clarkson's inequality

1228. If $0\leq x\leq\frac{\pi}{2}$ then:

$$\arcsin(\cos^2 x)\leq\frac{2\cos^2 x}{1+\sin x}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\sin^{-1}(\cos^2 x) \le \frac{2\cos^2 x}{1+\sin x}$$

$$\Leftrightarrow \sin^{-1}(1-\sin^2 x) \le 2(1-\sin x) \Leftrightarrow \sin^{-1}(1-t^2) \le 2(1-t)$$

$$\text{Let } f(t) = \sin^{-1}(1-t^2) - 2(1-t)$$

$$f'(t) = \frac{-2t}{\sqrt{1-(1-t^2)^2}} + 2 = \frac{-2t}{\sqrt{(2-t^2)t^2}} + 2 = 2\left[1 - \frac{1}{\sqrt{2-t^2}}\right] > 0 \text{ for } 0 < t < 1$$

$$\Rightarrow f(t) \text{ increases on } [0,1]$$

$$\Rightarrow f(t) \le f(1) = 0 \text{ for } 0 \le t \le 1 \Rightarrow \sin^{-1}(1-t^2) \le 2(1-t) \text{ for } 0 \le t \le 1$$

Solution 2 by David Chatarasvili-Georgia

$$\text{Consider the function: } f(x) = \arcsin(\cos^2 x) \cdot \frac{2\cos^2 x}{1+\sin x}$$

$$x \in \left[0, \frac{\pi}{2}\right]$$

$$f'(x) = \frac{-2\cos x \sin x}{\sqrt{1-\cos^4 x}} - \frac{-4\cos x \sin x\,(1+\sin x) - \cos x\,(2\cos^2 x)}{(1+\sin x)^2} =$$

$$= \frac{2\cos x\,(2\sin x\,(1+\sin x) + \cos^2 x)}{(1+\sin x)^2} - \frac{2\sin x \cdot \cos x}{\sqrt{1-\cos^4 x}} =$$

$$= \frac{2\cos(2\sin x + 2\sin^2 x + 1 - \sin^2 x)}{(1+\sin x)^2} - \frac{2\sin x \cdot \cos x}{\sqrt{1-\cos^4 x}} =$$

$$= 2\cos x\left(1 - \frac{\sin x}{\sqrt{1-\cos^4 x}}\right) = \frac{2\cos x\left(\sqrt{1-\cos^4 x} - \sin x\right)}{\sqrt{1-\cos^4 x}}$$

$$f'(x) = 0 \Rightarrow \left[\begin{array}{l} \cos x = 0 \\ \sqrt{1-\cos^4 x} = \sin x \quad (2) \end{array}\right.$$

$$(1) \Rightarrow x = \frac{\pi}{2} + \pi k, k \in \mathbb{Z}, x \in \left[0, \frac{\pi}{2}\right] \Rightarrow x = \frac{\pi}{2}$$

$$(2) \Leftrightarrow 1 - \cos^4 x = \sin x\left(\sin x \ge 0; x \in \left[0, \frac{\pi}{2}\right]\right)$$

$$1 - (1-\sin^2 x)^2 = 1 - (1 - 2\sin^2 x + \sin^4 x) = 2\sin^2 x - \sin^4 x =$$

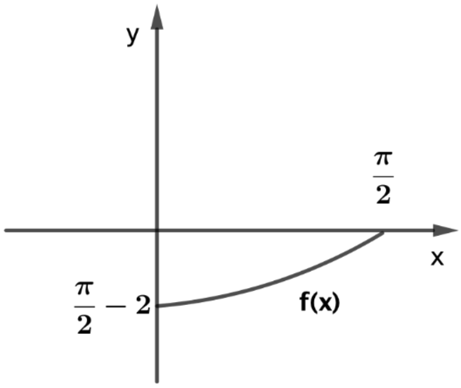
$$= \sin^2 x \Rightarrow \sin^2 x - \sin^4 x = 0 \Rightarrow \sin^2 x\,(1-\sin^2 x) = 0 \Rightarrow$$

$$\Rightarrow \sin^2 x \cdot \cos^2 x = 0 \Rightarrow \left[\begin{array}{l} \sin x = 0 \\ \cos x = 0 \end{array}\right. \Rightarrow \sin x = 0$$

$$x = \pi k, k \in \mathbb{Z}; x \in \left[0, \frac{\pi}{2}\right] \Rightarrow x = 0$$

$$\begin{cases} f'(x) = 0 \Rightarrow x = 0, x = \frac{\pi}{2} \\ f'(x) \geq 0; \forall x \in \left[0, \frac{\pi}{2}\right] \end{cases} \Rightarrow f(x) \text{ strictly increasing function}$$

$$f(0) = \arcsin 1 - \frac{2}{1} = \frac{\pi}{2} - 2 < 0, \qquad f\left(\frac{\pi}{2}\right) = \arcsin 0 - 0 = 0$$



$$\max_{x \in \left[0, \frac{\pi}{2}\right]} f(x) = f\left(\frac{\pi}{2}\right) = 0 \Rightarrow \forall x \in \left[0, \frac{\pi}{2}\right] f(x) \leq 0 \Rightarrow \arcsin(\cos^2 x) - \frac{2\cos^2 x}{1 + \sin x} \leq 0$$

$$\Rightarrow \arcsin(6\cos^2 x) \leq \frac{2\cos^2 x}{1 + \sin x}; \forall x \in \left[0, \frac{\pi}{2}\right]$$

1229. If $a, b, c > 0$,$4\left(\frac{2}{a} + b^2\right) \geq 7 + 5e^{\left(2c + \frac{1}{a^2} - 3\right)}$ and

$$27\left(\frac{a}{2c + a} + \frac{b}{2a + b} + \frac{c}{2b + c}\right) = 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 21, \text{then } a, b, c = ?$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 21 \overset{?}{\geq} 27\left(\frac{a}{2c + a} + \frac{b}{2a + b} + \frac{c}{2b + c}\right)$$

$$\Leftrightarrow 2\sum_{cyc} \frac{a}{b} + 21 \overset{?}{\geq} 27 \sum_{cyc} \frac{b + 2a - 2a}{2a + b} \Leftrightarrow \sum_{cyc} \frac{a}{b} + 27 \sum_{cyc} \frac{a}{2a + b} \overset{?}{\geq} 30$$

$$\Leftrightarrow \sum_{cyc} \frac{a}{b} + 27 \sum_{cyc} \frac{\frac{a}{b}}{2.\frac{a}{b} + 1} \overset{?}{\geq} 30$$

$$\Leftrightarrow \sum_{cyc} x + 27 \sum_{cyc} \frac{x}{2x + 1} \overset{?}{\geq} 30 \ \left(x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}\right)$$

$$\Leftrightarrow 4xyz\sum_{cyc}x+2\sum_{cyc}(x^2y+xy^2)+48xyz+\sum_{cyc}x^2\overset{?}{\geq}4\sum_{cyc}xy+16\sum_{cyc}x+15$$

$$\Leftrightarrow 4xyz\sum_{cyc}x+2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-6xyz+48xyz+\sum_{cyc}x^2\overset{?}{\geq}$$

$$4\sum_{cyc}xy+16\sum_{cyc}x+15$$

$$\Leftrightarrow 2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)+\sum_{cyc}x^2+27\overset{?}{\geq}4\sum_{cyc}xy+12\sum_{cyc}x\ (\because xyz=1)$$

$$\Leftrightarrow 2\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x-2\right)+\sum_{cyc}x^2+27\overset{?}{\geq}12\sum_{cyc}x$$

$$\Leftrightarrow\left(\left(\sum_{cyc}x\right)^2-\sum_{cyc}x^2\right)\left(\sum_{cyc}x-2\right)+\sum_{cyc}x^2+27\overset{?}{\geq}12\sum_{cyc}x$$

$$\Leftrightarrow\left(\sum_{cyc}x\right)^2\left(\sum_{cyc}x-2\right)+27\underset{(*)}{\overset{?}{\geq}}12\sum_{cyc}x+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x-3\right)$$

$$\text{Now,}\left(\sum_{cyc}xy\right)^2\geq 3xyz\left(\sum_{cyc}x\right)\overset{xyz=1}{=}3\sum_{cyc}x\Rightarrow\sum_{cyc}xy\geq\sqrt{3\sum_{cyc}x}$$

$$\therefore\sum_{cyc}x^2=\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy\leq\left(\sum_{cyc}x\right)^2-2.\sqrt{3\sum_{cyc}x}\ \text{and}$$

$$\because\sum_{cyc}x-3=\sum_{cyc}\frac{a}{b}-3\overset{A-G}{\geq}0,\therefore 12\sum_{cyc}x+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x-3\right)$$

$$\leq 12\sum_{cyc}x+\left(\left(\sum_{cyc}x\right)^2-2.\sqrt{3\sum_{cyc}x}\right)\left(\sum_{cyc}x-3\right)\overset{?}{\leq}\left(\sum_{cyc}x\right)^2\left(\sum_{cyc}x-2\right)+27$$

$$\Leftrightarrow 27+\frac{p^4}{9}\left(\frac{p^2}{3}-2\right)\overset{?}{\geq}12.\frac{p^2}{3}+\left(\frac{p^4}{9}-2p\right)\left(\frac{p^2}{3}-3\right)\left(\text{where } p=\sqrt{3\sum_{cyc}x}\right)$$

$$\Leftrightarrow\frac{p^4+6p^3-12p^2-18p+81}{27}\overset{?}{\geq}0\Leftrightarrow\frac{(p-3)^2(p+3)(p+9)}{27}\overset{?}{\geq}0\to\text{true}$$

$$\because p=\sqrt{3\sum_{cyc}x}=\sqrt{3\sum_{cyc}\frac{a}{b}}\overset{A-G}{\geq}3\Rightarrow(*)\text{ is true}$$

$$\therefore \boxed{2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+21\geq 27\left(\frac{a}{2c+a}+\frac{b}{2a+b}+\frac{c}{2b+c}\right),''='' \text{ iff } x=y=z \text{ and } \sqrt{3\sum_{cyc}\frac{a}{b}}=3}$$

$$\to (1)\ \text{Now, } \sqrt{3\sum_{cyc}\frac{a}{b}} \overset{A-G}{\geq} 3\ ('' = '' \text{ iff } a=b=c) \Rightarrow \sqrt{3\sum_{cyc}\frac{a}{b}}=3 \text{ iff } a=b=c$$

$$\text{and also } x=y=z \text{ iff } \frac{a}{b}=\frac{b}{c}=\frac{c}{a} \Rightarrow \text{iff } a=b=c \to (2)$$

$$\text{So, (1) and (2)} \Rightarrow 27\left(\frac{a}{2c+a}+\frac{b}{2a+b}+\frac{c}{2b+c}\right)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+21$$

$$\Rightarrow \boxed{\boxed{a=b=c}}$$

$$\text{Let } f(m)=e^m-1-m-\frac{m^2}{2}\ \forall\, m\geq 0 \text{ and then}: f'(m)=e^m-1-m\geq 0$$

$$\Rightarrow f(m) \text{ is } \uparrow \text{ on } [0,\infty) \Rightarrow f(m)\geq f(0)=1-1 \therefore e^m\geq 1+m+\frac{m^2}{2}\ \forall\, m\geq 0 \to (\bullet)$$

$$\text{Via } (\bullet), 7+5e^{\left(2a+\frac{1}{a^2}-3\right)}-4\left(\frac{2}{a}+a^2\right)\geq$$

$$7+5\left(1+\left(2a+\frac{1}{a^2}-3\right)+\left(2a+\frac{1}{a^2}-3\right)^2\right)-4\left(\frac{2}{a}+a^2\right)$$

$$=\frac{16a^6-50a^5+42a^4+12a^3-25a^2+5}{a^4}$$

$$=\frac{(a-1)^2\left((16a^2+14a+2)(a-1)^2+3\right)}{a^4}\geq 0 \because a>0$$

$$\therefore \boxed{4\left(\frac{2}{a}+a^2\right)\leq 7+5e^{\left(2a+\frac{1}{a^2}-3\right)}},''='' \text{ iff } 2a+\frac{1}{a^2}-3=0$$

$$\Rightarrow \text{iff } \frac{(2a+1)(a-1)^2}{a^2}=0 \Rightarrow \boxed{''='' \text{ iff } a=1} \to (3)$$

$$\text{Now}, \because a=b=c \therefore 4\left(\frac{2}{a}+b^2\right)\geq 7+5e^{\left(2c+\frac{1}{a^2}-3\right)} \Rightarrow$$

$$4\left(\frac{2}{a}+a^2\right)\geq 7+5e^{\left(2a+\frac{1}{a^2}-3\right)}, \text{but via (3)}, 4\left(\frac{2}{a}+a^2\right)\leq 7+5e^{\left(2a+\frac{1}{a^2}-3\right)}$$

$$\therefore 4\left(\frac{2}{a}+a^2\right)=7+5e^{\left(2a+\frac{1}{a^2}-3\right)} \text{ and via (3)}, \boxed{\boxed{a=1}} \therefore a=b=c=1\ (ans)$$

1230. $\left.\begin{matrix}\text{If } x>0\\ n\in\{1,2,3,\ldots\}\end{matrix}\right\}$, then the following relationship holds:

$$\left(\frac{x}{x+1}\right)^x+2\left(\frac{x+1}{x+2}\right)^{x+1}+\cdots+n\left(\frac{x+n}{x+n+1}\right)^{x+n}>\frac{n(n+1)}{2e}$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(m)=\ln m-1+\frac{1}{m}\ \forall\, m>0 \text{ and then}: f'(m)=\frac{m-1}{m^2}$$

$\boxed{\text{Case 1}}$ $m \le 1$ and then : $f'(m) \le 0\ \forall\ m \le 1 \Rightarrow f(m)$ is $\downarrow \forall\ m \in (0,1]$

$$\Rightarrow f(m) \ge f(1) = 0$$

$\boxed{\text{Case 2}}$ $m \ge 1$ and then : $f'(m) \ge 0\ \forall\ m \ge 1 \Rightarrow f(m)$ is $\uparrow \forall\ n \in [1,\infty)$

$$\Rightarrow f(m) \ge f(1) = 0 \therefore \text{combining both cases}, \ln m \ge 1 - \frac{1}{m}\ \forall\ m > 0, ''='' \text{ iff } m = 1$$

$$(x+n)\ln\left(\frac{x+n}{x+n+1}\right) > (x+n)\left(1 - \frac{x+n+1}{x+n}\right)$$

$$\left(\text{strict inequality} \because \frac{x+n}{x+n+1} \ne 1\right) \Rightarrow \ln\left(\frac{x+n}{x+n+1}\right)^{x+n} > -1$$

$$\Rightarrow \left(\frac{x+n}{x+n+1}\right)^{x+n} > \frac{1}{e}\ \forall\ x > 0 \text{ and } \forall\ n \in \{0,1,2,3,\dots\}$$

$$\therefore \left(\frac{x}{x+1}\right)^{x}, \left(\frac{x+1}{x+2}\right)^{x+1}, \dots, \left(\frac{x+n}{x+n+1}\right)^{x+n} > \frac{1}{e}$$

$$\Rightarrow \left(\frac{x}{x+1}\right)^{x} + 2\left(\frac{x+1}{x+2}\right)^{x+1} + \dots + n\left(\frac{x+n}{x+n+1}\right)^{x+n} > \frac{1}{e} + \frac{2}{e} + \dots + \frac{n}{e} = \frac{n(n+1)}{2e}$$

(QED)

1231. If $n \in \{3,4,5,\dots\}$, then the following relationship holds :

$$n^n < (n!)^2 < n^n.\left(\frac{n+1}{3}\right)^{n-1}$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via Stirling's approximation}, n^n < (n!)^2 < n^n.\left(\frac{n+1}{3}\right)^{n-1}$$

$$\Leftrightarrow n^n < 2\pi n\left(\frac{n}{e}\right)^{2n} < n^n.\left(\frac{n+1}{3}\right)^{n-1}$$

$$\text{Now}, n^n < 2\pi n\left(\frac{n}{e}\right)^{2n} \Leftrightarrow 2\pi n\left(\frac{n}{e^2}\right)^{n} > 1 \Leftrightarrow \ln(2\pi) + \ln(n) + n(\ln(n) - 2) > 1$$

$$\Leftrightarrow \ln(2\pi) + (n+1)\ln(n) - 2n \overset{(i)}{>} 0$$

Let $f(n) = \ln(2\pi) + (n+1)\ln(n) - 2n$ be a continuous function and then :

$$f'(n) = \ln(n) + \frac{1}{n} - 1 \text{ and } f''(n) = \frac{n-1}{n^2}$$

$\boxed{\text{Case 1}}$ $n \le 1$ and then : $f''(n) \le 0\ \forall\ n \le 1 \Rightarrow f'(n)$ is $\downarrow \forall\ n \in (0,1]$

$$\Rightarrow f'(n) \ge f'(1) = 0$$

$\boxed{\text{Case 2}}$ $n \ge 1$ and then : $f''(n) \ge 0\ \forall\ n \ge 1 \Rightarrow f'(n)$ is $\uparrow \forall\ n \in [1,\infty)$,

$$\Rightarrow f'(n) \ge f'(1) = 0 \therefore \forall\ n > 0, f'(n) \ge 0\ ('' ='' \text{ iff } n = 1) \Rightarrow \forall\ n \ge 3, f'(n) > 0$$

$$\Rightarrow \forall\ n \ge 3, f(n) \ge f(3) = \ln(2\pi) + 4\ln(3) - 6 \approx 0.2323 > 0 \Rightarrow \text{(i) is true}$$

$$\Rightarrow \forall\ n \ge 3, n^n < 2\pi n\left(\frac{n}{e}\right)^{2n} \Rightarrow \forall\ n \in \{3,4,5,\dots\}, \boxed{(n!)^2 > n^n}$$

$$\text{Again}, 2\pi n\left(\frac{n}{e}\right)^{2n} < n^n.\left(\frac{n+1}{3}\right)^{n-1} \Leftrightarrow 2\pi n\left(\frac{n}{e^2}\right)^{n} < \left(\frac{n+1}{3}\right)^{n-1}$$

$$\Leftrightarrow \ln(2\pi)+\ln(n)+n\ln(n)-2n<(n-1)\ln\left(\frac{n+1}{3}\right)$$

$$\Leftrightarrow (n-1)\ln\left(\frac{n+1}{3}\right)-\ln(2\pi)-(n+1)\ln(n)+2n\overset{(ii)}{>}0$$

Let $F(n)=(n-1)\ln\left(\frac{n+1}{3}\right)-\ln(2\pi)-(n+1)\ln(n)+2n$ be *a* continuous function *and* then : $F'(n)=\ln\left(\frac{n+1}{3}\right)-\ln(n)+\frac{n-1}{n+1}-\frac{1}{n}+1$ *and*

$$F''(n)=\frac{2n^2+n+1}{n^2(n+1)^2}>0\ \forall\, n\ge 3\Rightarrow F'(n)\text{ is }\uparrow\ \forall\, n\ge 3$$

$$\Rightarrow F'(n)\ge F'(3)\approx 0.3557>0\Rightarrow F(n)\text{ is }\uparrow\ \forall\, n\ge 3\Rightarrow F(n)\ge F(3)\approx 0.343>0$$

$$\Rightarrow \text{(ii) is true}\Rightarrow \forall\, n\ge 3,\ 2\pi n\left(\frac{n}{e}\right)^{2n}<n^n.\left(\frac{n+1}{3}\right)^{n-1}\Rightarrow \forall\, n\in\{3,4,5,\dots\},$$

$$\boxed{(n!)^2<n^n.\left(\frac{n+1}{3}\right)^{n-1}}\ \therefore\ n^n<(n!)^2<n^n.\left(\frac{n+1}{3}\right)^{n-1}\quad \forall\, n\in\{3,4,5,\dots\}\ \text{(QED)}$$

1232. If $a\ge b\ge c>0$, *then* :

$$108\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\le 189+5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^3$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=\frac{b^2-a^2}{ab}+\frac{c^2-b^2}{bc}+\frac{a^2-c^2}{ac}$$

$$=\frac{c(b+a)(b-a)+c^2(a-b)+ab(a-b)}{abc}=\frac{(b-a)\left(bc+ac-c^2-ab\right)}{abc}$$

$$=\frac{(b-a)\left(b(c-a)-c(c-a)\right)}{abc}=\frac{(b-a)(c-a)(b-c)}{abc}=\frac{(a-b)(b-c)(a-c)}{abc}$$

$$\ge 0\ (\because a\ge b\ge c>0)\Rightarrow x+y+z\ge\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ \left(x=\frac{b}{a},y=\frac{c}{b},z=\frac{a}{c}\right)$$

$$\Rightarrow \sum_{cyc}x\ge\sum_{cyc}xy\to(1)\ (\because xyz=1)\ \textit{and also},\left(\sum_{cyc}xy\right)^2\ge 3xyz\left(\sum_{cyc}x\right)$$

$$\Rightarrow \sum_{cyc}xy\ge\sqrt{3\sum_{cyc}x}\to(2)\ (\because xyz=1)$$

$$\text{Now, }189+5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)^3-108\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)=189+5\left(\sum_{cyc}x\right)^3-108\sum_{cyc}xy$$

$$\overset{\text{via }(1)}{\ge}\ 189+5\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x\right)^2-108\sum_{cyc}xy$$

$$= 189 + \left(\sum_{cyc} xy\right)\left(5\left(\sum_{cyc} x\right)^2 - 45 - 63\right) \overset{\text{via (1) and (2)}}{\geq}$$

$$189 + \sqrt{3\sum_{cyc} x} \cdot \left(5\left(\sum_{cyc} x\right)^2 - 45\right) - 63\left(\sum_{cyc} x\right)$$

$$\left(\because \sum_{cyc} x \overset{A-G}{\geq} 3\sqrt[3]{xyz} \overset{xyz=1}{=} 3 \Rightarrow 5\left(\sum_{cyc} x\right)^2 - 45 \geq 0\right)$$

$$= 189 + t.\left(\frac{5t^4}{9} - 45\right) - 63.\frac{t^2}{3}\left(\text{where } t = \sqrt{3\sum_{cyc} x}\right)$$

$$= \frac{5t^5 - 189t^2 - 405t + 1701}{9}$$

$$= \frac{(t-3)\left((t-3)(5t^3 + 30t^2 + 135t + 351) + 486\right)}{9} \geq 0$$

$$\left(\because \sum_{cyc} x \overset{A-G}{\geq} 3\sqrt[3]{xyz} \overset{xyz=1}{=} 3 \Rightarrow t = \sqrt{3\sum_{cyc} x} \geq \sqrt{3.3} \Rightarrow t \geq 3\right)$$

$$\therefore 108\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq 189 + 5\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)^3$$

$$\forall a \geq b \geq c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \frac{a}{b} + \frac{b}{c} + \frac{c}{a}$. **By AM − GM inequality, we have** $x \geq 3$.

Since $\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) = \frac{(a-b)(a-c)(b-c)}{abc} \geq 0,$

$$\text{then } \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \geq x, \text{ and}$$

$$189 + 5\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right)^3 \geq 189 + 5x^3 = 108x + (x-3)(5x^2 + 15x - 63)$$

$$\overset{x \geq 3}{\geq} 108x = 108\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right),$$

which completes the proof. Equality holds iff $a = b = c$.

Solution 3 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Let $x := \frac{b}{a}, \ y := \frac{c}{b}, \ z := \frac{a}{c}$ **and** $p := x + y + z, \ q := xy + yz + zx, \ r := xyz = 1.$

The desired inequality is equivalent to

$$108q \le 189 + 5p^3.$$

By AM − GM inequality, we have $p \ge 3\sqrt[3]{r} = 3$.

Also by Schur's inequality, we have $4pq \le p^3 + 9r$, then

$$108q \le \frac{27(p^3+9)}{p} = 189 + 5p^3 - \frac{(p-3)^2(5p^2+3p-27)}{p} \overset{p\ge 3}{\le} 189 + 5p^3,$$

which completes the proof. Equality holds iff $a = b = c$.

1233. If $x, y, z \ge 0$ then:

$$\sqrt{x^2+z^2+xz} + \sqrt{y^2+z^2+yz\sqrt{3}} \ge \sqrt{x^2+y^2}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let be $\omega \in \mathbb{C}, \omega^3 = 1, \omega \ne 1$.

$$x^2+z^2+xz = |x+\omega z|^2, \qquad y^2+z^2+yz\sqrt{3} = |iy-\omega z|^2$$

$$\sqrt{x^2+z^2+xz} + \sqrt{y^2+z^2+yz\sqrt{3}} = |x+\omega z| + |iy-\omega z| \ge$$

$$\ge |x+\omega z + iy - \omega z| = |x+iy| = \sqrt{x^2+y^2}$$

Equality holds for $x = y = z = 0$.

Solution 2 by proposer

Let be $M \in Int(\Delta ABC)$ such that:

$$m(\sphericalangle AMB) = 90°, m(\sphericalangle BMC) = 150°, m(\sphericalangle CMA) = 120°$$

$$\begin{cases} AC^2 = x^2+z^2-2xz\cos 120° = x^2+z^2+xz \\ BC^2 = y^2+z^2-2yz\cos 150° = y^2+z^2+yz\sqrt{3} \Rightarrow \\ AB^2 = x^2+y^2 \end{cases}$$

$$\Rightarrow \begin{cases} AC = \sqrt{x^2+z^2+xz} \\ BC = \sqrt{y^2+z^2+yz\sqrt{3}}, \\ AB = \sqrt{x^2+y^2} \end{cases} \quad AC + BC \ge AB \Rightarrow$$

$$\sqrt{x^2+z^2+xz} + \sqrt{y^2+z^2+yz\sqrt{3}} \ge \sqrt{x^2+y^2}$$

Equality holds for $x = y = z = 0$.

1234. If $x \in \left(0, \frac{\pi}{4}\right)$, then : $2\sin x + \tan x < 3x + (\ln x).\ln(1-x)$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Soumava Chakraborty-Kolkata-India

$$\forall x \in \left(0, \frac{\pi}{4}\right), \tan x \overset{?}{<} \frac{4x}{3} \Leftrightarrow 3\sin x \overset{?}{<} 4x\cos x$$

$$\Leftrightarrow \frac{3}{2}.2\sin x\cos x \overset{?}{<} 2x.2\cos^2 x \Leftrightarrow \frac{3}{2}.\sin 2x \overset{?}{<} 2x.(1+\cos 2x)$$

$$\Leftrightarrow \boxed{2y(1+\cos y) \underset{(\bullet)}{\overset{?}{>}} 3\sin y} \left(0 < y = 2x < \frac{\pi}{2}\right)$$

$$\text{Let } F(y) = \sin y - y + \frac{y^3}{6} - \frac{y^5}{120}\ \forall y \in \left[0, \frac{\pi}{2}\right)$$

$$\therefore F'(y) = \cos y - \frac{y^4}{24} + \frac{y^2}{2} - 1 \textit{ and } F''(y) = -\left(\sin y - \left(y - \frac{y^3}{6}\right)\right)$$

$$\text{Let } P(y) = \sin y - y + \frac{y^3}{6}\ \forall y \in \left[0, \frac{\pi}{2}\right) \therefore P'(y) = \cos y + \frac{y^2}{2} - 1 \textit{ and}$$

$$P''(y) = y - \sin y \geq 0 \Rightarrow P'(y) \text{ is } \uparrow \text{ on } \left[0, \frac{\pi}{2}\right) \Rightarrow P'(y) \geq P'(0) = 0$$

$$\Rightarrow P(y) \text{ is } \uparrow \text{ on } \left[0, \frac{\pi}{2}\right) \Rightarrow P(y) \geq P(0) = 0 \Rightarrow \sin y \geq y - \frac{y^3}{6}\ \forall y \in \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow F''(y) \leq 0 \Rightarrow F'(y) \text{ is } \downarrow \text{ on } \left[0, \frac{\pi}{2}\right) \Rightarrow F'(y) \leq F'(0) = 0 \Rightarrow F(y) \text{ is } \downarrow \text{ on } \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow F(y) \leq F(0) = 0 \Rightarrow \sin y \leq y - \frac{y^3}{6} + \frac{y^5}{120}\ \forall y \in \left[0, \frac{\pi}{2}\right)$$

$$\therefore \forall y \in \left(0, \frac{\pi}{2}\right), \boxed{\sin y \overset{(1)}{>} y - \frac{y^3}{6}} \textit{ and } \boxed{\sin y \overset{(2)}{<} y - \frac{y^3}{6} + \frac{y^5}{120}}$$

$$\therefore \textit{via } (2), \text{RHS of } (\bullet) < 3y - \frac{y^3}{2} + \frac{y^5}{40} \overset{?}{<} 2y(1+\cos y) \Leftrightarrow \boxed{\cos y \underset{(\bullet\bullet)}{\overset{?}{>}} \frac{1}{2} - \frac{y^2}{4} + \frac{y^4}{80}}$$

$$\text{Let } f(y) = \cos y - \frac{1}{2} + \frac{y^2}{4} - \frac{y^4}{80}\ \forall y \in \left(0, \frac{\pi}{2}\right]$$

$$\therefore f'(y) = -\sin y - \frac{y^3}{20} + \frac{y}{2} \overset{\text{via }(1)}{<} -y + \frac{y^3}{6} - \frac{y^3}{20} + \frac{y}{2} = -\frac{y}{2} + \frac{7y^3}{60} = -\frac{y}{2}\left(\frac{30-7y^2}{30}\right)$$

$$< 0 \left(\because 30 - 7y^2 \geq 30 - \frac{7\pi^2}{4} > 0\right) \Rightarrow f(y) \text{ is } \downarrow \text{ on } \left(0, \frac{\pi}{2}\right] \Rightarrow f(y) \geq f\left(\frac{\pi}{2}\right)$$

$$= -\frac{1}{2} + \frac{\pi^2}{16} - \frac{\pi^4}{16.80} \approx .0407 > 0 \Rightarrow \cos y - \frac{1}{2} + \frac{y^2}{4} - \frac{y^4}{80} > 0\ \forall y \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true } \therefore \forall x \in \left(0, \frac{\pi}{4}\right), \tan x < \frac{4x}{3} \Rightarrow 2\sin x + \tan x < 2x + \frac{4x}{3}$$

$$= \frac{10x}{3} \overset{?}{<} 3x + (\ln x).\ln(1-x) \Leftrightarrow \boxed{(\ln x).\ln(1-x) \underset{(*)}{\overset{?}{>}} \frac{x}{3}}$$

Let $g(x)=-\ln(1-x)-\left(x+\frac{x^2}{2}+\frac{x^3}{3}\right)\ \forall\, x\in\left[0,\frac{\pi}{4}\right)\ \therefore\ g'(x)=\frac{x^3}{1-x}\ge 0\Rightarrow g(x)$

is $\uparrow$ on $\left[0,\frac{\pi}{4}\right)\Rightarrow g(x)\ge g(0)=0\Rightarrow \boxed{-\ln(1-x)\overset{(i)}{>}x+\frac{x^2}{2}+\frac{x^3}{3}}\ \forall\, x\in\left(0,\frac{\pi}{4}\right)$

Let $h(x)=-\ln x-\left(1-x+\frac{(1-x)^2}{2}\right)\ \forall\, x\in(0,1]\ \therefore\ h'(x)=-\left(x+\frac{1}{x}-2\right)\le 0$

$\Rightarrow h(x)$ is $\downarrow$ on $(0,1]\Rightarrow h(x)\ge h(1)=0\Rightarrow -\ln x\ge 1-x+\frac{(1-x)^2}{2}\ \forall\, x\in(0,1]$

$\therefore\ \boxed{-\ln x\overset{(ii)}{>}1-x+\frac{(1-x)^2}{2}}\ \forall\, x\in\left(0,\frac{\pi}{4}\right)\ \therefore\ (i).(ii)\Rightarrow(\ln x).\ln(1-x)$

$$=(-\ln x).(-\ln(1-x))>\left(x+\frac{x^2}{2}+\frac{x^3}{3}\right)\left(1-x+\frac{(1-x)^2}{2}\right)\overset{?}{>}\frac{x}{3}$$

$$\Leftrightarrow \boxed{\frac{x(2x^4-5x^3-15x+14)}{12}\underbrace{\overset{?}{>}}_{(**)}0}$$

Now, $2x^4-5x^3-15x+14=\frac{5x-4}{625}(250x^3-425x^2-340x-2147)+\frac{162}{625}$

$$\because\ x<\frac{\pi}{4}<\frac{4}{5}\ \therefore\ 250x^3-425x^2-340x-2147$$

$<250x^2.\frac{4}{5}-425x^2-340x-2147=-225x^2-340x-2147<0$ and

$\because\ 5x-4<0\ \therefore\ \frac{5x-4}{625}(250x^3-425x^2-340x-2147)+\frac{162}{625}>0+\frac{162}{625}>0$

$\Rightarrow(**)\Rightarrow(*)$ is true $\therefore\ 2\sin x+\tan x<3x+(\ln x).\ln(1-x)\ \forall\, x\in\left(0,\frac{\pi}{4}\right)$ (QED)

1235.

If $n\in\mathbb{N}^*-\{1\}$, then the following relationship holds:

$$\binom{2n}{n}\ge 1+n^2+\binom{2n}{n-2}$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$\text{If } n=2,\binom{2n}{n}-\left(1+n^2+\binom{2n}{n-2}\right)=\binom{4}{2}-\left(1+4+\binom{4}{0}\right)=0$$

$$\Rightarrow\binom{2n}{n}=1+n^2+\binom{2n}{n-2}\text{ for } n=2$$

$$\text{If } n=3,\binom{2n}{n}-\left(1+n^2+\binom{2n}{n-2}\right)=\binom{6}{3}-\left(1+9+\binom{6}{1}\right)=20-10-6=4$$

$$\Rightarrow\binom{2n}{n}>1+n^2+\binom{2n}{n-2}\text{ for } n=3$$

$$\text{If } n=4,\binom{2n}{n}-\left(1+n^2+\binom{2n}{n-2}\right)=\binom{8}{4}-\left(1+16+\binom{8}{2}\right)=70-17-28$$

$$= 25 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 4$$

$$\text{If } n = 5, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{10}{5} - \left(1 + 25 + \binom{10}{3}\right)$$

$$= 252 - 26 - 120 = 106 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 5$$

$$\text{If } n = 6, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{12}{6} - \left(1 + 36 + \binom{12}{4}\right)$$

$$= 924 - 37 - 495 > 0 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 6$$

$$\text{If } n = 7, \binom{2n}{n} - \left(1 + n^2 + \binom{2n}{n-2}\right) = \binom{14}{7} - \left(1 + 49 + \binom{14}{5}\right)$$

$$= 3432 - 50 - 2002 > 0 \Rightarrow \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n = 7$$

$$\text{We now consider } n \ge 8; n \in \mathbb{N}^* \text{ and then : } \boxed{\binom{2n}{n} - \binom{2n}{n-2}}$$

$$= \frac{1.2.3 \ldots n.(n+1)(n+2)(n+3)(n+4)\ldots 2n}{(1.2.3\ldots n)(1.2.3\ldots n)} - \frac{1.2.3\ldots n.(n+1)(n+2)(n+3)(n+4)\ldots 2n}{\big(1.2.3\ldots(n-2)\big)\big(1.2.3\ldots n(n+1)(n+2)\big)}$$

$$= \frac{(n+1)(n+2)(n+3)(n+4)\ldots 2n}{1.2.3\ldots(n-2)(n-1)n} - \frac{(n+3)(n+4)\ldots 2n}{1.2.3\ldots(n-2)}$$

$$= \frac{(n+3)(n+4)\ldots 2n}{1.2.3\ldots(n-2)} \cdot \left(\frac{(n+1)(n+2)}{n(n-1)} - 1\right) = \frac{(n+3)(n+4)\ldots 2n}{1.2.3\ldots(n-2)} \cdot \frac{4n+2}{n^2-n}$$

$$= \frac{2n}{n-2} \cdot \frac{n+3}{1} \cdot \frac{n+4}{2} \cdot \frac{n+5}{3} \cdot \frac{4n+2}{n^2-n} \cdot \frac{(n+6)(n+7)\ldots(2n-1)}{4.5\ldots(n-3)}$$

$$= \boxed{\frac{2n}{n-2} \cdot \frac{n+3}{1} \cdot \frac{n+4}{2} \cdot \frac{n+5}{3} \cdot \frac{4n+2}{n^2-n} \cdot \prod_{k=6}^{n-1} \frac{n+k}{k-2}}$$

$$> \frac{2n}{n-2} \cdot \frac{n+3}{1} \cdot \frac{n+4}{2} \cdot \frac{n+5}{3} \cdot \frac{4n+2}{n^2-n} \left(\because \frac{n+k}{k-2} > 1 \ \forall\, k \in \{6.7\ldots(n-1)\}\right)$$

$$\boxed{\overset{?}{>} 1 + n^2} \Leftrightarrow \boxed{\boxed{n^5 + 59n^4 + 203n^3 + 343n^2 + 114n \overset{?}{>} 0}} \to \text{true}$$

$$\therefore \binom{2n}{n} > 1 + n^2 + \binom{2n}{n-2} \text{ for } n \ge 8; n \in \mathbb{N}^* \therefore \text{combining } \textit{all} \text{ cases},$$

$$\binom{2n}{n} \ge 1 + n^2 + \binom{2n}{n-2} \ \forall\, n \in \mathbb{N}^* - \{1\}, \text{''}=\text{''} \text{ iff } n = 2 \text{ (QED)}$$

1236.

$$\textbf{If } x > 0, \textbf{then :}$$

$$\pi\big(1 + (\arctan x)(\operatorname{arccot} x)^{-1}\big) > 8 + 2(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x)$$

Proposed by Rovsen Pirguliyev-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } F(x)=\pi\left(1+(\arctan x)(\operatorname{arccot} x)^{-1}\right)-8$$
$$-2\left(\ln^2\arctan x+\ln^2\operatorname{arccot} x\right)\ \forall\, x>0$$
$$\therefore F'(x)=$$
$$-\frac{4(\operatorname{arccot} x)^2(\arctan x)\ln(\arctan x)-4(\arctan x)^2(\operatorname{arccot} x)\ln(\operatorname{arccot} x)+\pi\begin{pmatrix}\operatorname{arccot} x\\-\arctan x\end{pmatrix}}{(x^2+1)(\arctan x)^2(\operatorname{arccot} x)^2}$$
$$\frac{4(\arctan x)^2(\operatorname{arccot} x)\ln(\operatorname{arccot} x)-4(\operatorname{arccot} x)^2(\arctan x)\ln(\arctan x)-\pi\begin{pmatrix}\operatorname{arccot} x\\-\arctan x\end{pmatrix}}{(\arctan x)^2(\operatorname{arccot} x)^2}$$
$$=4\left(\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}-\frac{\ln(\arctan x)}{\arctan x}-\frac{\pi}{4}\frac{(\operatorname{arccot} x-\arctan x)}{(\arctan x)^2(\operatorname{arccot} x)^2}\right)$$
$$=4\left(\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}-\frac{\ln(\arctan x)}{\arctan x}-\frac{1}{2}\frac{(\operatorname{arccot} x+\arctan x)(\operatorname{arccot} x-\arctan x)}{(\arctan x)^2(\operatorname{arccot} x)^2}\right)$$
$$=4\left(\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}-\frac{\ln(\arctan x)}{\arctan x}-\frac{1}{2}\left(\frac{1}{(\arctan x)^2}-\frac{1}{(\operatorname{arccot} x)^2}\right)\right)$$
$$=4\left(\left(\frac{1}{2(\operatorname{arccot} x)^2}+\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}\right)-\left(\frac{1}{2(\arctan x)^2}+\frac{\ln(\arctan x)}{\arctan x}\right)\right)$$
$$\Rightarrow F'(x)=\frac{4}{x^2+1}\cdot\begin{pmatrix}\left(\frac{1}{2(\operatorname{arccot} x)^2}+\frac{\ln(\operatorname{arccot} x)}{\operatorname{arccot} x}\right)\\-\left(\frac{1}{2(\arctan x)^2}+\frac{\ln(\arctan x)}{\arctan x}\right)\end{pmatrix}$$
$$\text{Let } f(m)=\frac{1}{2m^2}+\frac{\ln m}{m}\ \forall\, m>0\ \therefore f'(m)=\frac{m-1-m\ln m}{m^3}=\frac{m-1}{m^3}-\frac{\ln m}{m^2}$$
$$\le\frac{m-1}{m^3}-\frac{1-\frac{1}{m}}{m^2}\left(\because \ln m\ge 1-\frac{1}{m}\ \forall\, m>0\right)=0\ \therefore\ \frac{m-1-m\ln m}{m^3}\le 0\to(1)$$
$$\boxed{\text{Case 1}}\ 0<x\le 1 \text{ and then}: \operatorname{arccot} x\ge\arctan x \text{ and } F'(x)$$
$$=\frac{4}{x^2+1}\cdot\left(f(\operatorname{arccot} x)-f(\arctan x)\right)\overset{MVT}{=}$$
$$\boxed{\frac{4}{x^2+1}\cdot(\operatorname{arccot} x-\arctan x)\cdot\frac{\xi-1-\xi\ln\xi}{\xi^3}}\overset{\text{via }(1)}{\le}0$$
$$\Rightarrow F(x)\text{ is }\downarrow\text{ on }(0,1]\Rightarrow F(x)\ge F(1)=\pi\left(1+\frac{16}{\pi^2}\right)-8-4\ln^2\frac{\pi}{4}\approx .0011>0$$
$$\Rightarrow\pi\left(1+(\arctan x)(\operatorname{arccot} x)^{-1}\right)>8+2\left(\ln^2\arctan x+\ln^2\operatorname{arccot} x\right)$$
$$\boxed{\text{Case 2}}\ x\ge 1 \text{ and then}: \arctan x\ge\operatorname{arccot} x \text{ and } F'(x)$$
$$=-\frac{4}{x^2+1}\cdot\left(f(\arctan x)-f(\operatorname{arccot} x)\right)\overset{MVT}{=}$$
$$\boxed{-\frac{4}{x^2+1}\cdot(\arctan x-\operatorname{arccot} x)\cdot\frac{\xi-1-\xi\ln\xi}{\xi^3}}\overset{\text{via }(1)}{\ge}0$$
$$\Rightarrow F(x)\text{ is }\uparrow\text{ on }[1,\infty)\Rightarrow F(x)\ge F(1)=\pi\left(1+\frac{16}{\pi^2}\right)-8-4\ln^2\frac{\pi}{4}\approx .0011>0$$
$$\Rightarrow\pi\left(1+(\arctan x)(\operatorname{arccot} x)^{-1}\right)>8+2\left(\ln^2\arctan x+\ln^2\operatorname{arccot} x\right)$$

$$\therefore \text{combining both cases}, \pi\left(1+(\arctan x)(\operatorname{arccot} x)^{-1}\right)$$
$$> 8 + 2\left(\ln^2 \arctan x + \ln^2 \operatorname{arccot} x\right)\ \forall\, x > 0\ (\text{QED})$$

1237. If $x, y, z > 1, m, n, p \geq 1$ then:

$$(x-1)^{m+n+p} + x^{m+n} + x^{n+p} + x^{p+m} + 1 \leq x^{m+n+p} + x^m + x^n + x^p$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

We first show that: $x^m + 1 \leq (x+1)^m \quad \forall x > 0, m \geq 1$

Let $f(x) = x^m + 1 - (x+1)^m, x \geq 0 \Rightarrow f'(x) = mx^{m-1} - m(x+1)^{m-1} \leq 0, x > 0$

$\Rightarrow f(x)$ decreases on $[0, \infty)$

Thus, $f(x) \leq f(0) \quad \forall x > 0 \Rightarrow x^m + 1 - (x+1)^m \leq 0; \forall x > 0$

$\Rightarrow (x-1)^m + 1 \leq x^m; \forall x > 1 \Rightarrow (x-1)^m \leq x^m - 1; \forall x > 1, m \geq 1$

Thus, for $x > 1, m, n, p \geq 1$

$$(x-1)^m(x-1)^n(x-1)^p \leq (x^m-1)(x^n-1)(x^p-1)$$

$$\Rightarrow (x-1)^{m+n+p} \leq x^{m+n+p} - x^{m+n} - x^{m+p} - x^{n+p} + x^m + x^n + x^p - 1$$

$$\Rightarrow (x-1)^{m+n+p} + x^{m+n} + x^{m+p} + x^{n+p} + 1 \leq x^{m+n+p} + x^m + x^n + x^p$$

1238. If $x, y, z > 0$ then: $e(x^x + y^y + z^z) \geq 3\sqrt[3]{e^{x+y+z}}$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $f(x) = x(\ln x - 1) + 1; \forall x > 0$

$f'(x) = \ln x - 1 + x\left(\frac{1}{x}\right) = \ln x \Rightarrow f'(x) < 0$ for $0 < x < 1, > 0$ if $x > 1$

Thus, for $0 < x \leq 1, f(1) \leq f(x)$ and for $x \geq 1, f(1) \leq f(x)$

$\Rightarrow f(x) \geq f(1) = 0; \forall x > 0 \Rightarrow x(\ln x - 1) + 1 \geq 0; \forall x > 0$

$\Rightarrow x \ln x \geq x - 1 \Rightarrow \ln(x^x) \geq x - 1 \Rightarrow x^x \geq e^{x-1}; \forall x > 0 \Rightarrow ex^x \geq e^x$

Thus, for $x, y, z > 0$

$$e(x^x + y^y + z^z) \geq e^x + e^y + e^z \geq 3(e^x e^y e^z)^{\frac{1}{3}} = 3(e^{x+y+z})^{\frac{1}{3}}$$

$$\Rightarrow e(x^x + y^y + z^z) \geq 3\left(e^{\frac{x+y+z}{3}}\right)$$

Equality when $x = y = z = 1$.

Solution 2 by Hikmat Mammadov-Azerbaijan

The inequality of arithmetic and geometric means gives $\sqrt[3]{e^x e^y e^z} \leq \frac{e^x+e^y+e^z}{3}$

So, $3\sqrt[3]{e^{x+y+z}} \leq e^x + e^y + e^z$

The study of variations of the function $f: x \to 1 + x\ln(x) - x$

Shows that the minimum of this function on, $\mathbb{R}^{+*}$ is 0 (for $x = 1$)

So, $x \leq 1 + x\ln(x) - x, y \leq 1 + y\ln(y) - y$ and $z \leq 1 + z\ln(z) - z$

So, $e^x \leq e \cdot x^x, e^y \leq e \cdot y^y$ and $e^z \leq e \cdot z^z$

So, $e^x + e^y + e^z \leq e \cdot (x^x + y^y + z^z)$

Finally: $3 \cdot \sqrt[3]{e^{x+y+z}} \leq e \cdot (x^x + y^y + z^z)$

Solution 3 by Khaled Abd Imouti-Damascus-Syria

$$e \cdot (x^x + y^y + z^z) \geq 3 \cdot \left(e^{\frac{x+y+z}{3}}\right), e \cdot \left(e^{x\ln(x)} + e^{y\ln(y)} + e^{z\ln(z)}\right) \geq 3\left(e^{\frac{x+y+z}{3}}\right)$$

$$e^{1+x\ln(x)} + e^{1+y\ln(y)} + e^{1+3\ln(z)} \geq 3 \cdot \left(e^{\frac{x+y+z}{3}}\right)$$

$$f(x) = 1 + x\ln(x), 'f(x) = \ln(x) + 1 \Rightarrow ''f(x) = \frac{1}{x} > 0$$

$$l_1 \geq 3 \cdot e^{1+\left(\frac{x+y+z}{3}\right)\ln\left(\frac{x+y+z}{3}\right)}, \qquad l_1 \geq e \cdot \left(\frac{x+y+z}{3}\right)$$

$$1 + x\ln(x) \overset{?}{\geq} x, \qquad x\ln(x) \geq x - 1, \qquad x\ln(x) - x + 1 \overset{?}{\geq} 0$$

$$f(x) = x\ln(x) - x + 1, x > 0, \lim_{x\to 0^+} f(x) = 1, \lim_{x\to+\infty} f(x) = +\infty$$

$$'f(x) = \ln(x) + 1 - 1 = \ln(x)$$

x	0 $\qquad$ 1 $\qquad$ $+\infty$
$f'(x)$	$- - - - - - - 0 + + + + + + + +$
$f(x)$	$\searrow$ $\nearrow$

Solution 4 by Pin Reak Smey-Indonesia

We have $ex^x \geq e^x \Leftrightarrow \ln e + \ln x^x \geq x \Leftrightarrow 1 + x\ln x - x \geq 0$

Let $f(x) = 1 + x\ln x - x,\ f'(x) = \ln x$

x	$0 \qquad\qquad 1 \qquad\qquad +\infty$
$f'(x)$	$- - - - - - - - 0 + + + + + + + + +$
$f(x)$	$\searrow \qquad \nearrow$

$$\Rightarrow f(x) \geq 0 \Rightarrow f(1) = 0$$

$$ex^x \geq e^x, \forall x > 0 \Rightarrow \sum_{cyc} ex^x \geq \sum_{cyc} e^x$$

But $\sum_{cyc} e^x \geq 3\sqrt[3]{e^{x+y+z}}$, thus $\sum_{cyc} ex^x \geq 3\sqrt[3]{e^{x+y+z}}$

1239. If $x, y, z \in \mathbb{R}$ then:

$$2(5 - x - y - z)^2 + 14(x^2 + y^2 + z^2) \geq 35$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban-Romania

$$x^2 + y^2 + z^2 \overset{CBS}{\geq} \frac{(x + y + z)^2}{3}$$

$$S = x + y + z$$

$$\Rightarrow 2 \cdot (5 - S)^2 + 14\sum_{cyc} x^2 \geq 2(5 - S)^2 + \frac{14S^2}{3} \geq 35 \Rightarrow$$

$$\Rightarrow 50 - 20S + 2S^2 + \frac{14S^2}{3} \geq 35 \Rightarrow 150 - 60S + 6S^2 + 14S^2 \geq 105 \Rightarrow$$

$$\Rightarrow 20S^2 - 60S + 45 \geq 0 | : 5$$

$$\Rightarrow 4S^2 - 12S + 9 \geq 0 \Rightarrow (2S - 3)^2 \geq 0$$

true, $(\forall) S \in \mathbb{R}$

Then $2 \cdot (S - x - y - z)^2 + 14(x^2 + y^2 + z^2) \geq 35, (\forall) x, y, z \in \mathbb{R}$

Solution 2 by George Florin Șerban-Romania

The second method:

$$2 \cdot (5 - x - y - z)^2 + 14(x^2 + y^2 + z^2) \geq 35$$

$$2\cdot(25-10\sum_{cyc}x+(x+y+z)^2+14\sum_{cyc}x^2\geq 35$$

$$50-20\sum_{cyc}x+2\sum_{cyc}x^2+4\sum_{cyc}xy+14\sum_{cyc}x^2-35\geq 0$$

$$16\sum_{cyc}x^2+4\sum_{cyc}xy-20\sum_{cyc}x+15\geq 0$$

$$\left(x+2y-\frac{3}{2}\right)^2+\left(y+2z-\frac{3}{2}\right)^2+\left(z+2x-\frac{3}{2}\right)^2+$$

$$+11\left(x-\frac{1}{2}\right)^2+11\left(y-\frac{1}{2}\right)^2+11\left(z-\frac{1}{2}\right)^2=$$

$$=x^2+4y^2+\frac{9}{4}+4xy-6y-3x+y^2+4z^2+\frac{9}{4}+$$

$$+11x^2-11x+\frac{11}{4}+11y^2-11y+\frac{11}{y}+11z^2-11z+\frac{11}{4}=$$

$$=16\sum_{cyc}x^2+4\sum_{cyc}xy-20\sum_{cyc}x+15\geq 0$$

true, because $\left(x+2y-\frac{3}{2}\right)^2\geq 0,(\forall)x,y\in\mathbb{R}$

$$\left(y+2z-\frac{3}{2}\right)^2\geq 0;(\forall)y,z\in\mathbb{R},\left(z+2x-\frac{3}{3}\right)^2\geq 0$$

$(\forall)y\in\mathbb{R},\left(z-\frac{1}{2}\right)^2\geq 0;(\forall)z\in\mathbb{R}$. Equality is if $x-\frac{1}{2}=y-\frac{1}{2}=z-\frac{1}{2}=0$

$$x+2y-\frac{3}{2}=y+2z-\frac{3}{2}=z+2x-\frac{3}{2}=0\Rightarrow x=y=z=\frac{1}{2}\in\mathbb{R}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$2(5-x-y-z)^2+14(x^2+y^2+z^2)\geq 35$$

$$\Leftrightarrow 2[25+x^2+y^2+z^2-10x-10y-10z+2xy+2yz+2zx]+$$

$$+14x^2+14y^2+14z^2\geq 35$$

$$\Leftrightarrow 16x^2+16y^2+16z^2+4xy+4yz+4zx-$$

$$-20x+20y-20z+15\geq 0$$

$$\Leftrightarrow 3(4x^2-4x+1)+3(4y^2-4y+1)+3(4z^2-4z+1)+$$

$$+2(1+x^2+y^2+2xy-2x-2y)+$$

$$+2(1+y^2+z^2+2yz-2y-2z)+$$

$$+2(1+z^2+x^2+2zx-2z-2x)\geq 0 \Leftrightarrow 3[(2x-1)^2+(2y-1)^2+(2z-1)^2]+$$

$$+2[(1-x-y)^2+(1-y-z)^2+(1-z-x)^2]\geq 0$$

which is true and equality when $x=y=z=\frac{1}{2}$.

Solution 4 by Khaled Abd Imouti-Damascus-Syria

$$2\left(5-(x+y+z)\right)^2+14(x^2+y^2+z^2)\overset{?}{\geq} 35$$

$$2[25-10(x+y+z)+(x^2+y^2+z^2)+2(xy+yz+zx)]+14(x^2+y^2+z^2)\overset{?}{\geq} 35$$

$$16\underbrace{(x^2+y^2+z^2)}_{A}+4\underbrace{(xy+yz+zx)}_{B}-20\underbrace{(x+y+z)}_{\gamma}+15\overset{?}{\geq} 0$$

$$P_1=16A+4B-20\gamma+15\overset{?}{\geq} 0$$

but: $(x+y+z)^2=x^2+y^2+z^2+2(xy+yz+zx)$

$\gamma^2=A+2B$, but $B\leq A$ (Cauchy Schwarz inequality)

$$\gamma^2\leq 3A, A\geq\frac{1}{3}\gamma^2, B=\frac{\gamma^2-A}{2}$$

$$P_1=16A+4\left(\frac{\gamma^2-A}{2}\right)-20\gamma+15\overset{?}{\geq} 0$$

$$P_1=16A+2(\gamma^2-A)-20\gamma+15\overset{?}{\geq} 0$$

$$P_1=2\gamma^2-20\gamma+14A+15\overset{?}{\geq} 0$$

$$P_1=2\left(\gamma^2-10\gamma+7A+\frac{15}{2}\right)\overset{?}{\geq} 0$$

$A\geq\frac{1}{3}\gamma^2$. So: $l_1\geq 2\left(\gamma^2-10\gamma+\frac{7}{3}\gamma^2+\frac{15}{2}\right)$, $l_1\geq 2\left(\frac{10}{3}\gamma^2-10\gamma+\frac{15}{2}\right)$

$\Delta=100-4\left(\frac{10}{3}\right)\left(\frac{15}{2}\right)=100-100=0$.So: $\frac{10}{3}\gamma^2-10\gamma+\frac{15}{2}\geq 0, \forall\gamma\in\mathbb{R}$

and then: $l_1\geq 0$

1240.

If $x\in\left(0,\frac{\pi}{4}\right)$ then: $2\sin x+\tan x<3x+\ln x\cdot\ln(1-x)$

Proposed by Khaled Abd Imouti-Damascus-Syria

Solution by Mohamed Amine Ben Ajiba-Tanger-Morooco

Using the well known inequality, $lnt \le t-1, \forall t>0$,
we have $-lnx > 1-x > 0$ and,

$$-ln(1-x) > x > 0, \text{ then } lnx.ln(1-x) > (1-x)x, \ \forall x \in \left(0,\frac{\pi}{4}\right).$$

So it suffices to prove that $f(x) = 4x - x^2 - 2\sin x - \tan x \ge 0, \ \forall x \in \left[0,\frac{\pi}{4}\right]$.

We have

$$f''(x) = -2(1-\sin x) - 2(1+\tan^2 x)\tan x < 0, \text{then } f \text{ is concave on } \left[0,\frac{\pi}{4}\right], \text{and}$$

$$f(x) \ge min\left\{f(0), f\left(\frac{\pi}{4}\right)\right\} = min\left\{0, \pi - \frac{\pi^2}{16} - \sqrt{2} - 1\right\} = 0, \forall x \in \left[0,\frac{\pi}{4}\right],$$

which completes the proof.

1241. If $n \in \{2,3,\dots\}$, then prove that

$$1 + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).\sqrt[3]{\frac{n(n+1)^2}{4}} > 2\sqrt[n]{n!}$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By CBS inequality, we have

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \ge \frac{n^2}{1+2+\cdots+n} = \frac{2n}{n+1}$$

Then

$$1 + \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).\sqrt[3]{\frac{n(n+1)^2}{4}} \ge 1 + \frac{2n}{n+1}.\sqrt[3]{\frac{n(n+1)^2}{4}}$$

$$= 1 + n.\sqrt[3]{\frac{2n}{n+1}} \overset{n>1}{>} 1 + n =$$

$$= 2 \cdot \frac{1+2+\cdots+n}{n} \overset{AM-GM}{>} 2.\sqrt[n]{1.2\dots n} = 2.\sqrt[n]{n!}.$$

1242.

Let $a,b,c \in [0,2023]$. Find the max and min value of the expression :

$$P = a^3 + b^3 + c^3 - 2024abc$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM − GM inequality, we have

$$P \geq 3abc - 2024abc = -2021abc \geq -2021.2023^3,$$

then the min value of P is -2021.2023^3, reached at $a = b = c = 2023$.

Now, let $f(a) = P$, we have $f''(a) = 6a \geq 0$, then f is convex on $[0, 2023]$, and

$$P \leq max\{f(0), f(2023)\} = max\{b^3 + c^3, 2023^3 + b^3 + c^3 - 2024.2023bc\}.$$

Also, we have $b^3 + c^3 \leq 2.2023^3$, and if $g(b) = 2023^3 + b^3 + c^3 - 2024.2023bc$,

then g is convex on $[0, 2023]$, and $g(b) \leq max\{g(0), g(2023)\}$

$$= max\{2023^3 + c^3, 2.2023^3 + c^3 - 2024.2023^2c\} \overset{c\leq 2023}{\leq} 2.2023^3,$$

then the max value of P is 2.2023^3, reached at $a = b = 2023, c = 0$ and permutations.

1243. **If $a, b, c > 0$ then:**

$$\sqrt[3]{4a} + \sqrt[3]{9b} + \sqrt[3]{25c} \leq \sqrt[3]{100(a + b + c)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by George Florin Șerban – Romania

$$\sqrt[3]{4a} + \sqrt[3]{9b} + \sqrt[3]{25c} = \sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{a} + \sqrt[3]{3} \cdot \sqrt[3]{3} \cdot \sqrt[3]{b} + \sqrt[3]{5} \cdot \sqrt[3]{5} \cdot \sqrt[3]{c} \overset{Holder}{\leq}$$

$$\leq \left(\sqrt[3]{2}^3 + \sqrt[3]{3}^3 + \sqrt[3]{5}^3\right)^{\frac{1}{3}} \cdot \left(\sqrt[3]{2}^3 + \sqrt[3]{3}^3 + \sqrt[3]{5}^3\right)^{\frac{1}{3}} \cdot \left(\sqrt[3]{a}^3 + \sqrt[3]{b}^3 + \sqrt[3]{c}^3\right)^{\frac{1}{3}} =$$

$$= (2 + 3 + 5)^{\frac{1}{3}} \cdot (2 + 3 + 5)^{\frac{1}{3}} \cdot (a + b + c)^{\frac{1}{3}} =$$

$$= \sqrt[3]{10 \cdot 10 \cdot (a + b + c)} = \sqrt[3]{100(a + b + c)}$$

then

$$\sqrt[3]{4a} + \sqrt[3]{9b} + \sqrt[3]{25c} \leq \sqrt[3]{100(a + b + c)}$$

$(\forall)a, b, c > 0$, true

Solution 2 by Michel Sterghiou – Greece

$$\sqrt[3]{4a} + \sqrt[3]{9b} + \sqrt[3]{25c} \leq \sqrt[3]{100(a + b + c)} \quad \textbf{(1)}$$

Let $\alpha = 2x, b = 3y, c = 5z$ (1) → $2\sqrt[3]{x} + 3\sqrt[3]{y} + 5\sqrt[3]{z} \leq \sqrt[3]{100(2x + 3y + 5z)}$ (2)

Using the fact that if $(t) = \sqrt[3]{t}$ $(0, +\infty)$ is concave we have by generalize Jensen

$$LHS\ (2) \le (2+3+5)\cdot\sqrt[3]{\frac{2x+3y+5z}{2+3+5}} = \sqrt[3]{1000\cdot\frac{a+b+c}{10}} = \sqrt[3]{100(a+b+c)} = RHS \text{ of } (2).$$

Equality for $a = 2, b = 3, c = 5$.

1244. If $a, b, c, d, e > 0$ then:

$$\frac{a+b+c}{d+e}+\frac{b+c+d}{e+a}+\frac{c+d+e}{a+b}+\frac{d+e+a}{b+c}+\frac{e+a+b}{c+d}\ge\frac{15}{2}$$

(Vasic's variant)

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\frac{a+b+c}{d+e}+1+\frac{b+c+d}{e+a}+1+\frac{c+d+e}{a+b}+1+$$

$$+\frac{d+e+a}{b+c}+1+\frac{e+a+b}{c+d}+1\ge\frac{15}{2}+5$$

$$\frac{a+b+c+d+e}{d+e}+\frac{a+b+c+d+e}{e+a}+\frac{a+b+c+d+e}{a+b}+$$

$$+\frac{a+b+c+d+e}{b+c}+\frac{a+b+c+d+e}{c+d}\ge\frac{15+10}{2}=\frac{25}{2}$$

$$(a+b+c+d+e)\left(\frac{1}{d+e}+\frac{1}{e+a}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}\right)\ge\frac{25}{2}$$

$$(a+b+c+d+e)\left(\frac{1}{d+e}+\frac{1}{e+a}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+d}\right)\ge$$

$$\overset{Bergstrom}{\ge}(a+b+c+d+e)\cdot\frac{(1+1+1+1+1)^2}{d+e+e+a+a+b+b+c+c+d}=$$

$$=\frac{25(a+b+c+d+e)}{2a+2b+2c+2d+2e}=\frac{25(a+b+c+d+e)}{2(a+b+c+d+e)}=\frac{25}{2}, \text{ true}$$

$$\Rightarrow\frac{a+b+c}{d+e}+\frac{b+c+d}{e+a}+\frac{c+d+e}{a+b}+\frac{d+e+a}{b+c}+\frac{e+a+b}{c+d}\ge\frac{15}{2}, (\forall)a,b,c,d,e>0$$

Equality is if $a = b = c = d = e$.

Solution 2 by Tapas Das-India

Let $(a+b+c+d+e) = x$

$$\therefore LHS = \frac{a+b+c}{x-(a+b+c)} + \frac{b+c+d}{x-(b+c+d)} + \frac{c+d+e}{x-(c+d+e)} +$$

$$+\frac{d+e+a}{x-(d+e+a)} + \frac{e+a+b}{x-(e+a+b)}$$

Let $f(p) = \frac{p}{x-p}, p > 0$

$$\therefore f'(p) = \frac{x}{(x-p)^2} \therefore f''(p) = \frac{2x}{(x-p)^3} > 0$$

$\therefore$ f is convex, using Jensen inequality

$$f(a+b+c) + f(b+c+d) + f(c+d+e) + f(d+e+a) + f(e+a+b) \geq$$

$$\geq 5f\left(\frac{3a+3b+3c+3d+3e}{5}\right) = 5f\left(\frac{3x}{5}\right) = 5 \cdot \frac{\frac{3x}{5}}{x - \frac{3x}{5}} = 5 \times \frac{3}{2} = \frac{15}{2}$$

Solution 3 by Sakthi Vel-India

$$\frac{a+b+c}{d+e} = \frac{a+b+c}{d+e} + \frac{d+e}{d+e} - 1 = \frac{a+b+c+d+e}{d+e} - 1$$

$$\sum_{cyc} \frac{a+b+c}{d+e} = (a+b+c+d+e)\sum_{cyc}\frac{1}{d+e} - 5$$

$$\geq (a+b+c+d+e)\frac{(1+1+1+1+1)^2}{2(a+b+c+d+e)} - 5 = \frac{25}{2} - 5 = \frac{15}{2}$$

1245. **If $a, b, c, \alpha \geq 1$ then:**

$$\frac{a}{c} + \frac{b}{a} + \frac{c}{b} + \alpha(ab+bc+ca) \geq (1+\alpha)(a+b+c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-India

For $x, y, \alpha \geq 1,\ x\left(\alpha - \frac{1}{y}\right)(y-1) \geq 0 \Rightarrow \alpha xy + \frac{x}{y} - x - \alpha x \geq 0$

$$\Rightarrow \frac{x}{y} + \alpha xy \geq (1+\alpha)x$$

$$\therefore$$

$$\frac{a}{c} + \alpha ac \geq (1+\alpha)a$$

$$\frac{b}{a}+\alpha ab\geq(1+\alpha)b$$

and $\frac{c}{b}+\alpha bc\geq(1+\alpha)c$

Adding above inequalities the desired inequality.

Solution 2 by Tapas Das-India

We need to show,

$$\frac{a}{c}+\frac{b}{a}+\frac{c}{b}+\alpha(ab+bc+ca)\geq(1+\alpha)(a+b+c)$$

$$\left[\frac{a}{c}+\alpha ac-(1+\alpha)a\right]+\left[\frac{b}{a}+\alpha ab-(a+\alpha)b\right]+\left[\frac{c}{b}+\alpha bc-(1+\alpha)c\right]\geq 0$$

or, $a\left[\frac{1}{c}+\alpha c-(1+\alpha)\right]+b\left[\frac{1}{a}+\alpha a-(1+\alpha)\right]+c\left[\frac{1}{b}+\alpha b-(1+\alpha)\right]\geq 0$

or, $a\left[\frac{(1-c)(1-c\alpha)}{c}\right]+b\left[\frac{(1-a)(1-a\alpha)}{a}\right]+c\left[\frac{(1-b)(1-b\alpha)}{b}\right]\geq 0$

or, $\frac{a}{c}(1-c)(1-c\alpha)+\frac{b}{a}(1-a)(1-a\alpha)+\frac{c}{b}(1-b)(1-b\alpha)\geq 0$

This is true. Since, $c\geq 1,\alpha\geq 1\therefore c\alpha\geq 1$

$$\therefore(1-c)\leq 0,(1-c\alpha)\leq 0$$

$\therefore(1-c)(1-c\alpha)\geq 0$ **(analog)**

1246.

If $x,y\in\left[0,\frac{\pi}{2}\right)$, then :

$$(\sin x+\sin y)(x+y)(\tan x+\tan y)\overset{(*)}{\leq}(\sin x+y)(x+\tan y)(\tan x+\sin y)$$
$$\overset{(**)}{\leq}(\sin x+\tan y)(x+y)(\tan x+\sin y)$$

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

If $x=y=0$, then : LHS of $(*)$ = RHS of $(*)=0$ and LHS of $(**)$
$=$ RHS of $(**)=0$

If $x=0,y\in\left(0,\frac{\pi}{2}\right)$, then : LHS of $(*)$ = RHS of $(*)=y\sin y\tan y$ and LHS of $(**)$
$=$ RHS of $(**)=y\sin y\tan y$

If $y=0, x\in\left(0,\frac{\pi}{2}\right)$, then : LHS of $(*)$ = RHS of $(*) = x\sin x\tan x$ *and* LHS of $(**)$

$= \text{RHS of } (**) = x\sin x\tan x$

We now consider $x, y\in\left(0,\frac{\pi}{2}\right)$ *and*

$$(\sin x+\sin y)(x+y)(\tan x+\tan y)\le(\sin x+y)(x+\tan y)(\tan x+\sin y)$$

$$\Leftrightarrow\frac{\tan x+\tan y}{\tan x+\sin y}-1\le\frac{(\sin x+y)(x+\tan y)}{(\sin x+\sin y)(x+y)}-1$$

$$\Leftrightarrow\frac{\tan y-y+y-\sin y}{\tan x+\sin y}\le$$

$$\frac{x\sin x+\sin x\tan y+xy+y\tan y-x\sin x-y\sin x-x\sin y-y\sin y}{(\sin x+\sin y)(x+y)}$$

$$\Leftrightarrow\frac{(\tan y-y)\sin x+x(y-\sin y)+y\big((\tan y-y)+(y-\sin y)\big)}{(\sin x+\sin y)(x+y)}$$

$$\ge\frac{(\tan y-y)+(y-\sin y)}{\tan x+\sin y}$$

$$\Leftrightarrow(\tan y-y)\left(\frac{y+\sin x}{(\sin x+\sin y)(x+y)}-\frac{1}{\tan x+\sin y}\right)$$

$$+(y-\sin y)\left(\frac{1}{\sin x+\sin y}-\frac{1}{\tan x+\sin y}\right)\ge 0\Leftrightarrow$$

$$\binom{\tan y}{-y}\left(\frac{\tan x\sin x+y\tan x+\sin x\sin y+y\sin y-x\sin x-x\sin y-y\sin x-y\sin y}{(\sin x+\sin y)(x+y)(\tan x+\sin y)}\right)$$

$$+(y-\sin y)\left(\frac{\tan x-\sin x}{(\sin x+\sin y)(\tan x+\sin y)}\right)\ge 0$$

$$\Leftrightarrow\binom{\tan y}{-y}\left(\frac{\sin x(\tan x-x)+y\big((\tan x-x)+(x-\sin x)\big)-\sin y(x-\sin x)}{(\sin x+\sin y)(x+y)(\tan x+\sin y)}\right)$$

$$+(y-\sin y)\left(\frac{\tan x-\sin x}{(\sin x+\sin y)(\tan x+\sin y)}\right)\ge 0$$

$$\Leftrightarrow(\tan y-y)\left(\frac{(\tan x-x)(\sin x+y)+(y-\sin y)(x-\sin x)}{(\sin x+\sin y)(x+y)(\tan x+\sin y)}\right)$$

$$+(y-\sin y)\left(\frac{\tan x-\sin x}{(\sin x+\sin y)(\tan x+\sin y)}\right)\ge 0$$

$\rightarrow$ true $\because \tan x>x>\sin x$ *and* $\tan y>y>\sin y\ \forall x,y\in\left(0,\frac{\pi}{2}\right)$

$$\therefore(\sin x+\sin y)(x+y)(\tan x+\tan y)<(\sin x+y)(x+\tan y)(\tan x+\sin y)$$

and combining *all* cases,

$$\boxed{(\sin x+\sin y)(x+y)(\tan x+\tan y)\le(\sin x+y)(x+\tan y)(\tan x+\sin y)\ \forall\, x,y\in\left[0,\frac{\pi}{2}\right)},$$

$$''=''\ \text{iff } (x=y=0) \text{ or } \left(x=0, y\in\left(0,\frac{\pi}{2}\right)\right) \text{ or } \left(y=0, x\in\left(0,\frac{\pi}{2}\right)\right)$$

Again, $\boxed{(\sin x+y)(x+\tan y)(\tan x+\sin y)\le(\sin x+\tan y)(x+y)(\tan x+\sin y)}$

$$\Leftrightarrow \frac{\sin x+\tan y}{\sin x+y}-1\ge\frac{x+\tan y}{x+y}-1\Leftrightarrow\frac{\tan y-y}{\sin x+y}\ge\frac{\tan y-y}{x+y}$$

$$\Leftrightarrow(\tan y-y)\left(\frac{x+y-\sin x-y}{(x+y)(\sin x+y)}\right)\ge 0\ \boxed{\Leftrightarrow\frac{(\tan y-y)(y-\sin y)}{(x+y)(\sin x+y)}\ge 0}\to\text{true}$$

$$\because \tan y>y>\sin y\ \ \forall\, x,y\in\left(0,\frac{\pi}{2}\right)$$

$$\therefore(\sin x+y)(x+\tan y)(\tan x+\sin y)<(\sin x+\tan y)(x+y)(\tan x+\sin y)$$

and combining *all* cases,

$$\boxed{(\sin x+y)(x+\tan y)(\tan x+\sin y)\le(\sin x+\tan y)(x+y)(\tan x+\sin y)\ \forall\, x,y\in\left[0,\frac{\pi}{2}\right)},$$

$$''=''\ \text{iff } (x=y=0) \text{ or } \left(x=0, y\in\left(0,\frac{\pi}{2}\right)\right) \text{ or } \left(y=0, x\in\left(0,\frac{\pi}{2}\right)\right)\ \text{(QED)}$$

1247. If $x,y,z>0, x+y+z=1$ and $\lambda\ge\frac{2}{5}$, then :

$$\sum_{cyc}\sqrt{2\lambda x^2+6\lambda xy+(2\lambda+1)y^2}\le\sqrt{10\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{2\lambda x^2+6\lambda xy+(2\lambda+1)y^2}\overset{?}{\le}\sum_{cyc}\left(\left(\frac{x+y}{2}\right).\sqrt{10\lambda+1}\right)$$

$$\Leftrightarrow y.\frac{(t_1+1)}{2}.\sqrt{10\lambda+1}+z.\frac{(t_2+1)}{2}.\sqrt{10\lambda+1}+x.\frac{(t_3+1)}{2}.\sqrt{10\lambda+1}$$

$$\overset{?}{\ge} y.\sqrt{2\lambda t_1^2+6\lambda t_1+2\lambda+1}+z.\sqrt{2\lambda t_2^2+6\lambda t_2+2\lambda+1}$$

$$+x.\sqrt{2\lambda t_3^2+6\lambda t_3+2\lambda+1}\ \left(t_1=\frac{x}{y}, t_2=\frac{y}{z}, t_3=\frac{z}{x}\right)$$

$$\Leftrightarrow y\left(\frac{(t_1+1)}{2}.\sqrt{10\lambda+1}-\sqrt{2\lambda t_1^2+6\lambda t_1+2\lambda+1}\right)$$

$$+z\left(\frac{(t_2+1)}{2}.\sqrt{10\lambda+1}-\sqrt{2\lambda t_2^2+6\lambda t_2+2\lambda+1}\right)$$

$$+x\left(\frac{(t_3+1)}{2}.\sqrt{10\lambda+1}-\sqrt{2\lambda t_3^2+6\lambda t_3+2\lambda+1}\right)\overset{?}{\underset{(*)}{\geq}}0$$

Let $f(t)=\frac{(t+1)}{2}.\sqrt{10\lambda+1}-\sqrt{2\lambda t^2+6\lambda t+2\lambda+1}\ \forall\, t>0$ and $\forall\,\lambda\geq\frac{2}{5}$ and

then : $f''(t)=\frac{\lambda(5\lambda-2)}{(2\lambda t^2+6\lambda t+2\lambda+1)^{\frac{3}{2}}}\overset{\lambda\geq\frac{2}{5}}{\geq}0\Rightarrow f(t)$ is convex $\therefore$ via

weighted Jensen's inequality, $\frac{y.f(t_1)+z.f(t_2)+x.f(t_3)}{x+y+z}\geq f\left(\frac{yt_1+zt_2+xt_3}{x+y+z}\right)$

$$\Rightarrow y.f(t_1)+z.f(t_2)+x.f(t_3)\geq f\left(\frac{y.\frac{x}{y}+z.\frac{y}{z}+x.\frac{z}{x}}{x+y+z}\right)\ (\because x+y+z=1)$$

$$\Rightarrow \text{LHS of }(*)\geq f(1)=\frac{(1+1)}{2}.\sqrt{10\lambda+1}-\sqrt{2\lambda.1^2+6\lambda.1+2\lambda+1}=0$$

$$\Rightarrow (*)\text{ is true}\therefore\sum_{cyc}\sqrt{2\lambda x^2+6\lambda xy+(2\lambda+1)y^2}\leq\sum_{cyc}\left(\left(\frac{x+y}{2}\right).\sqrt{10\lambda+1}\right)$$

$$=\sqrt{10\lambda+1}.\frac{2(x+y+z)}{2}\overset{x+y+z=1}{=}\sqrt{10\lambda+1},\text{ i.e.},\sum_{cyc}\sqrt{2\lambda x^2+6\lambda xy+(2\lambda+1)y^2}$$

$$\leq\sqrt{10\lambda+1}\ \forall\, x,y,z>0\,|\,x+y+z=1\text{ and }\lambda\geq\frac{2}{5},\text{ "="}\text{ iff }x=y=z=\frac{1}{3}\ (QED)$$

1248. Prove that:

$$n\in\{2,3,\dots\}\Rightarrow\sqrt[n+1]{1+(n+1)\sqrt{\binom{2n}{n}\left(2^2+\left(\frac{2^2}{2}\right)^2+\left(\frac{2^3}{3}\right)^2+\dots+\left(\frac{2^{n+1}}{n+1}\right)^2\right)}}>3$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $\binom{2n}{n}=\binom{n}{0}^2+\binom{n}{1}^2+\dots+\binom{n}{n}^2$, then by the Cauchy – Schwarz inequality, we have

$$\sqrt{\binom{2n}{n}\left(2^2+\left(\frac{2^2}{2}\right)^2+\left(\frac{2^3}{3}\right)^2+\dots+\left(\frac{2^{n+1}}{n+1}\right)^2\right)}=\sqrt{\sum_{k=0}^{n}\binom{n}{k}^2.\sum_{k=0}^{n}\left(\frac{2^{k+1}}{k+1}\right)^2}$$

$$\geq\sum_{k=0}^{n}2^{k+1}.\frac{\binom{n}{k}}{k+1}$$

$$=\sum_{k=0}^{n}2^{k+1}.\frac{\binom{n+1}{k+1}}{n+1}=\frac{1}{n+1}.\sum_{k=1}^{n+1}\binom{n+1}{k}.2^k.1^{n+1-k}=\frac{(2+1)^{n+1}-1}{n+1}=\frac{3^{n+1}-1}{n+1},$$

with equality if $\frac{k+1}{2^{k+1}}\binom{n}{k} = \text{constant}$,

$\forall k \in \{0,1,\dots,n\}$, which is not true for any $n \geq 2$.

Therefore

$$\sqrt[n+1]{1+(n+1)\sqrt{\binom{2n}{n}\left(2^2+\left(\frac{2^2}{2}\right)^2+\left(\frac{2^3}{3}\right)^2+\dots+\left(\frac{2^{n+1}}{n+1}\right)^2\right)}} > 3$$

1249. **Let $x, y \in \mathbb{R}$ such that $x + y = x^2 + y^2$.**

Find the maximum and the minimum value of $A = x^3 + y^3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $x + y = x^2 + y^2 \geq 0$ then $A = x^3 + y^3 = (x+y)\left[\left(x-\frac{y}{2}\right)^2+\frac{3y^2}{4}\right] \geq 0$,

so the minimum value of A is 0, for $x = y = 0$.

Now by AM − GM inequality, we have:

$$4(x+y)^3A = (x+y)^2.(x+y)^2.4(x^2-xy+y^2) \leq$$

$$\leq \left(\frac{(x+y)^2+(x+y)^2+4(x^2-xy+y^2)}{3}\right)^3$$

$= 8(x^2+y^2)^3$, then $A \leq 2$, so the maximum value of A is 2, for $x = y = 1$.

1250. **Let $x, y \geq 0$ such that $x + y = 1$.**

Find the maximum and the minimum value of $P = \sin(\sqrt{x}) + \sin(\sqrt{y})$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $t \to \sin t$ is increasing and concave on $[0,1]$

then by Jensen's inequality, we have

$$P=\sin(\sqrt{x})+\sin(\sqrt{y})\le 2\sin\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)\le 2\sin\left(\sqrt{\frac{x+y}{2}}\right)=2\sin\left(\frac{\sqrt{2}}{2}\right),$$

so the maximum value of P is $2\sin\left(\frac{\sqrt{2}}{2}\right)$, for $x=y=\frac{1}{2}$.

Also, by Jensen's inequality, we have

$$\sin(\sqrt{x})\overset{\sqrt{x}\ge x}{\ge}\sin(x)=\sin(x.1+y.0)\ge x\sin 1+y\sin 0=x\sin 1.$$

Similarly, we have $\sin(\sqrt{y})\ge y\sin 1$.

Then $P\ge(x+y)\sin 1=\sin 1$, so the minimum value of P is $\sin 1$,

for $x=1$ and $y=0$ or $x=0$ and $y=1$.

1251. **Let $x,y\in\mathbb{R}$. Find the maximum and the minimum value of**
$$A=\sin x+\sin y+\sin(x+y)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

If M is the maximum value of A for $x=x_0$ and $y=y_0$ then we have

$$A=-[\sin(-x)+\sin(-y)+\sin(-x-y)]\ge -M,\quad \forall x,y\in\mathbb{R},$$

so $-M$ is the minimum value of P for $x=-x_0$ and $y=-y_0$.

Now, we have

$$A=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)+\sin(x+y)=2\left[\cos\left(\frac{x-y}{2}\right)+\cos\left(\frac{x+y}{2}\right)\right]\sin\left(\frac{x+y}{2}\right)$$

$$=4\cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\cos\left(\frac{z}{2}\right),\quad \text{where } z=\pi-(x+y).$$

If $\cos\left(\frac{x}{2}\right),\cos\left(\frac{y}{2}\right),\cos\left(\frac{z}{2}\right)\le 0$, we have $A\le 0$. WLOG we assume that $\cos\left(\frac{z}{2}\right)\ge 0$.

We have $A=2\left[\cos\left(\frac{x-y}{2}\right)+\cos\left(\frac{x+y}{2}\right)\right]\cos\left(\frac{z}{2}\right)\le 2\left[1+\sin\left(\frac{z}{2}\right)\right]\cos\left(\frac{z}{2}\right)$

$$\overset{CBS}{\leq} 2\sqrt{(2+1)\left(\frac{1}{2}+\sin^2\left(\frac{z}{2}\right)\right)\cos\left(\frac{z}{2}\right)} \overset{AM-GM}{\leq} \sqrt{3}\left[\left(\frac{1}{2}+\sin^2\left(\frac{z}{2}\right)\right)+\cos^2\left(\frac{z}{2}\right)\right]=\frac{3\sqrt{3}}{2},$$

with equality for $\cos\left(\frac{x-y}{2}\right)=1,\ \ \sin^2\left(\frac{z}{2}\right)+\frac{1}{2}=\cos^2\left(\frac{z}{2}\right),\ \ \sin\left(\frac{z}{2}\right)\geq 0 \Leftrightarrow$

$x,y\equiv\frac{\pi}{3}(mod2\pi)$. So the maximum value of A is $\frac{3\sqrt{3}}{2}$ for $x,y\equiv\frac{\pi}{3}(mod2\pi)$ and

the minimum value of A is $-\frac{3\sqrt{3}}{2}$ for $x,y\equiv-\frac{\pi}{3}(mod2\pi)$.

1252. If $\left(a>0 \text{ and } A=a^2+a^4+\frac{1}{a^6}\right)$, then :

$$1620A\leq 3645+5\left(\frac{189+5A^3}{108}\right)^5$$

Proposed by Pavlos Trifon-Greece

Solution by Soumava Chakraborty-Kolkata-India

$$A\overset{?}{\leq}\frac{189+5A^3}{108}\Leftrightarrow 5A^3-108A+189\overset{?}{\geq}0$$
$$\Leftrightarrow (A-3)\big((A-3)(5A+30)+27\big)\overset{?}{\geq}0\to \text{true}\because A=a^2+a^4+\frac{1}{a^6}$$
$$\overset{A-G}{\geq}3\sqrt[3]{a^2.a^4.\frac{1}{a^6}}=3\therefore 1620A-\left(3645+5\left(\frac{189+5A^3}{108}\right)^5\right)$$
$$\leq 1620\left(\frac{189+5A^3}{108}\right)-\left(3645+5\left(\frac{189+5A^3}{108}\right)^5\right)\overset{?}{\leq}0$$
$$\Leftrightarrow t^5-324t+729\overset{?}{\geq}0\ \left(t=\frac{189+5A^3}{108}\right)$$
$$\Leftrightarrow (t-3)\left((t-3)(t^3+6t^2+27t+108)+81\right)\overset{?}{\geq}0\to\text{true}\because t=\frac{189+5A^3}{108}\geq A$$
$$=a^2+a^4+\frac{1}{a^6}\overset{A-G}{\geq}3\sqrt[3]{a^2.a^4.\frac{1}{a^6}}=3$$
$$\therefore 1620A\leq 3645+5\left(\frac{189+5A^3}{108}\right)^5,''='' \text{ iff } a=1\ (QED)$$

1253. If $0<a\leq b\leq c\leq\frac{1}{e}$ then: $a^{a\left(b^b-c^c\right)}.b^{b\left(c^c-a^a\right)}.c^{c\left(a^a-b^b\right)}\leq 1$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Let } x = a^a,\ \ y = b^b,\ \ z = c^c \text{ and } f(x) = x^x,\ \ x \in \left(0, \frac{1}{e}\right].$$

$$\text{We have } f'(x) = ln(xe).x^x \le 0,$$

$$\text{then } f \text{ is decreasing on } \left(0, \frac{1}{e}\right] \text{ and } 0 < z \le y \le x \le \lim_{x\to 0^+} f(x) = 1.$$

The problem becomes to prove that

$$x^{y-z}.y^{z-x}.z^{x-y} \le 1 \iff (y-z).lnx + (z-x).lny + (x-y).lnz \le 0,$$

$$\iff (y-z)(lnx - lny) \le (x-y)(lny - lnz) \iff (y-z).ln\left(\frac{x}{y}\right) \le (x-y).ln\left(\frac{y}{z}\right),$$

$$\text{Using the known inequality, } 1 - \frac{1}{t} \le lnt \le t - 1,\ \ \forall t > 0, \text{ we have}$$

$$(y-z).ln\left(\frac{x}{y}\right) \le (y-z).\left(\frac{x}{y} - 1\right) = (x-y)\left(1 - \frac{z}{y}\right) \le (x-y).ln\left(\frac{y}{z}\right),$$

which completes the proof. Equality holds iff $a = b = c$.

1254.

$$\text{If } x, y, z \ge -\frac{1}{2} \text{ such that } x + y + z = -\frac{3}{4} \text{ and}$$

$$\lambda \ge \frac{5}{4} \text{ then find the maximum value of}$$

$$P = \sum_{cyc} \frac{2x+1}{4x^2 + 4x + \lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\text{Since } 2x+1, 2y+1, 2z+1 \ge 0, \text{ then we have}$$

$$P = \sum_{cyc} \frac{2x+1}{\left[(2x+1)^2 + \frac{1}{4}\right] + \left(\lambda - \frac{5}{4}\right)} \overset{AM-GM}{\le} \sum_{cyc} \frac{2x+1}{(2x+1) + \left(\lambda - \frac{5}{4}\right)} = \sum_{cyc} \left(1 - \frac{\lambda - \frac{5}{4}}{2x + \lambda - \frac{1}{4}}\right)$$

$$= 3 - \left(\lambda - \frac{5}{4}\right)\sum_{cyc}\frac{1}{2x + \lambda - \frac{1}{4}} \overset{CBS}{\leq} 3 - \frac{9\left(\lambda - \frac{5}{4}\right)}{2(x + y + z) + 3\left(\lambda - \frac{1}{4}\right)} =$$

$$= 3 - \frac{9\left(\lambda - \frac{5}{4}\right)}{-\frac{3}{2} + 3\left(\lambda - \frac{1}{4}\right)} = \frac{6}{4\lambda - 3},$$

so the maximum value of P is $\frac{6}{4\lambda - 3}$, for $x = y = z = -\frac{1}{4}$.

1255. **If $x, y, z > 0, x + y^3 + z^5 \geq x^2 + y^4 + z^6$ then:**

$$x + y + z \leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Using the AM – GM inequality, we have

$$3 + x + y^3 + z^5 \geq 3 + x^2 + y^4 + z^6 = (1 + x^2) + \frac{1 + 3y^4}{4} + \frac{3 + y^4}{4} + \frac{1 + 5z^6}{6} + \frac{5 + z^6}{6}$$

$$\geq 2x + \sqrt[4]{1.(y^4)^3} + \sqrt[4]{1^3.y^4} + \sqrt[6]{1.(z^6)^5} + \sqrt[6]{1^5.z^6} = 2x + y^3 + y + z^5 + z.$$

Then $3 \geq x + y + z$. Equality holds iff $x = y = z = 1$.

1256. **If $a, b, c \in \mathbb{R}, 2^a + 2^b + 2^c \geq 3^a + 3^b + 3^c$ then:**

$$a + b + c \leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mohamed Amine Ben jiba-Tanger-Morocco

By using the Mean Value Theorem for $f(x) = e^{ax}$ on the interval $[ln2, ln3]$, there exists $c \in (ln2, ln3)$ such that

$$3^a - 2^a = f(ln3) - f(ln2) = ln\left(\frac{3}{2}\right)f'(c) = ln\left(\frac{3}{2}\right).ae^{ac} =$$

$$= ln\left(\frac{3}{2}\right).[a + a(e^{ac} - 1)] \geq ln\left(\frac{3}{2}\right).a,$$

because a and $e^{ac} - 1$ have the same sign for all $a \in \mathbb{R}$.

$$\Rightarrow\ 3^a - 2^a \geq ln\left(\frac{3}{2}\right).a\ \text{(and analogs)}$$

$$\Rightarrow 0 \ge (3^a - 2^a) + (3^b - 2^b) + (3^c - 2^c) \ge \ln\left(\frac{3}{2}\right).(a+b+c).$$

Therefore $a+b+c \le 0$. Equality holds iff $a=b=c=0$.

1257.

$$\text{If } \begin{cases} a, b > 0 \\ n \in \{2,3,\dots\} \end{cases}, \text{then :}$$

$$1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \dots + \frac{n(na+b)}{a+nb} > \frac{n(n+1)}{2(2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}}}$$

Proposed by Pavlos Trifon-Greece

Solution 1 by Soumava Chakraborty-Kolkata-India

$$(2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}} = (1^1.2^2.3^3.\dots.n^n)^{\frac{1}{1+2+3+\dots+n}} \overset{\text{Weighted GM > Weighted HM}}{>}$$

$$\frac{1+2+3+\dots+n}{\frac{1}{1}+\frac{2}{2}+\frac{3}{3}+\dots+\frac{n}{n}} \therefore (2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}} > \frac{n(n+1)}{2n} \to (1)$$

$$\text{Also, } 1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \dots + \frac{n(na+b)}{a+nb}$$

$$= 1 + \frac{a+2b+3a}{a+2b} + \frac{a+3b+8a}{a+3b} + \dots + \frac{a+nb+(n^2-1)a}{a+nb}$$

$$= \underbrace{1+1+1+\dots+1}_{n \text{ terms}} + \sum_{k=2}^{n} \frac{(k^2-1)a}{a+kb} = n + \sum_{k=2}^{n} \frac{(k^2-1)a}{a+kb}$$

$$\therefore 1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \dots + \frac{n(na+b)}{a+nb} > n \to (2)$$

$$\therefore \left((2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}}\right)\left(1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \dots + \frac{n(na+b)}{a+nb}\right)$$

$$\overset{\text{via (1) and (2)}}{>} \frac{n(n+1)}{2n}.n = \frac{n(n+1)}{2}$$

$$\Rightarrow 1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \dots + \frac{n(na+b)}{a+nb} > \frac{n(n+1)}{2(2^2.3^3.\dots.n^n)^{\frac{2}{n(n+1)}}}$$

$$\forall \begin{cases} a, b > 0 \\ n \in \{2,3,\dots\} \end{cases} \text{(QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$1 + \frac{2(2a+b)}{a+2b} + \frac{3(3a+b)}{a+3b} + \dots + \frac{n(na+b)}{a+nb} =$$

$$= 1 + \left(1 + \frac{3a}{a+2b}\right) + \left(1 + \frac{8a}{a+3b}\right) + \dots + \left(1 + \frac{(n^2-1)a}{a+nb}\right) > 1+1+1+\dots+1 = n.$$

So it suffices to prove that

$$(2^2.3^3\dots n^n)^{\frac{2}{n(n+1)}}\ge\frac{n+1}{2}\ \text{ or }\ 2.ln(2)+3.ln(3)+\cdots+n.ln(n)\ge\frac{n(n+1)}{2}.ln\left(\frac{n+1}{2}\right).$$

The function $f(x)=x.ln(x),\ x>0$, is convex on $(0,\infty)$, then by Jensen's inequality, we have

$$2ln(2)+3ln(3)+\cdots+nln(n)=\sum_{k=1}^{n}f(k)\ge nf\left(\frac{1}{n}\sum_{k=1}^{n}k\right)=nf\left(\frac{n+1}{2}\right)=\frac{n(n+1)}{2}.ln\left(\frac{n+1}{2}\right).$$

So the proof is complete.

1258.

If $a,b>0$ prove that

$$\left(\frac{2b}{b+2a}+\sqrt[3]{\left(\frac{b}{a}\right)^2}+\frac{4(a^2+1)}{3(1+ab)}\right)\cdot\sqrt[9]{\frac{a^2(b+2a)^2(9+2a^2+2b^2+5ab)^4}{b^4(a^2+1)^4}}\ge 9\sqrt[9]{3}$$

Proposed by Pavlos Trifon-Greece

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

By AM – GM inequality, we have

$$\frac{2b}{b+2a}+\sqrt[3]{\left(\frac{b}{a}\right)^2}+\frac{4(a^2+1)}{3(1+ab)}=2.\frac{b}{b+2a}+3.\frac{1}{3}\sqrt[3]{\left(\frac{b}{a}\right)^2}+4.\frac{a^2+1}{3(1+ab)}$$

$$\ge 9\sqrt[9]{\left(\frac{b}{b+2a}\right)^2\cdot\left(\frac{1}{3}\sqrt[3]{\left(\frac{b}{a}\right)^2}\right)^3\cdot\left(\frac{a^2+1}{3(1+ab)}\right)^4}$$

$$=9\sqrt[9]{\frac{3b^4(a^2+1)^4}{a^2(b+2a)^2(9+9ab)^4}}$$

$$=9\sqrt[9]{\frac{3b^4(a^2+1)^4}{a^2(b+2a)^2(9+4ab+5ab)^4}}\ge 9\sqrt[9]{\frac{3b^4(a^2+1)^4}{a^2(b+2a)^2(9+2a^2+2b^2+5ab)^4}},$$

$$\Rightarrow \left(\frac{2b}{b+2a}+\sqrt[3]{\left(\frac{b}{a}\right)^2}+\frac{4(a^2+1)}{3(1+ab)}\right)\cdot\sqrt[9]{\frac{a^2(b+2a)^2(9+2a^2+2b^2+5ab)^4}{b^4(a^2+1)^4}}\ge 9\sqrt[9]{3}.$$

Equality holds iff $a=b$.

1259. **If $x,y,z\ge -\frac{1}{2}$ such that $x+y+z=-\frac{3}{4}$ and $\lambda\ge \frac{5}{4}$ then find the maximum value**

$$P=\sum_{cyc}\frac{2x+1}{4x^2+4x+\lambda}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $2x+1,2y+1,2z+1\ge 0$, then we have

$$P=\sum_{cyc}\frac{2x+1}{\left[(2x+1)^2+\frac{1}{4}\right]+\left(\lambda-\frac{5}{4}\right)}\overset{AM-GM}{\le}\sum_{cyc}\frac{2x+1}{(2x+1)+\left(\lambda-\frac{5}{4}\right)}$$

$$=\sum_{cyc}\left(1-\frac{\lambda-\frac{5}{4}}{2x+\lambda-\frac{1}{4}}\right)$$

$$=3-\left(\lambda-\frac{5}{4}\right)\sum_{cyc}\frac{1}{2x+\lambda-\frac{1}{4}}\overset{CBS}{\le}3-\frac{9\left(\lambda-\frac{5}{4}\right)}{2(x+y+z)+3\left(\lambda-\frac{1}{4}\right)}=$$

$$=3-\frac{9\left(\lambda-\frac{5}{4}\right)}{-\frac{3}{2}+3\left(\lambda-\frac{1}{4}\right)}=\frac{6}{4\lambda-3},$$

so the maximum value of P is $\frac{6}{4\lambda-3}$, for $x=y=z=-\frac{1}{4}$.

1260. **If $x,y,z\in\mathbb{R}$ such that $3x+y+2z\ge 3$ and $-x+2y+4z\ge 5$, then find the $min(x+2y+4z)$**

Proposed by Neculai Stanciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $x+2y+4z=\frac{4}{7}(3x+y+2z)+\frac{5}{7}(-x+2y+4z)\ge\frac{4}{7}.3+\frac{5}{7}.5=\frac{37}{7}$,

so the minimum value of $x+2y+4z$ is $\frac{37}{7}$, for $x=\frac{1}{7}, y=\frac{18}{7}-2t, z=t, \ t\in\mathbb{R}$.

1261. If $a,b,c\in[0,1]$ then find:

$$min\Omega, \quad max\Omega, \quad \Omega=a^2+b^2+c^2-abc$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Let be $f:[0,1]X[0,1]X[0,1]\to\mathbb{R}, f(x,y,z)=x^2+y^2+z^2-xyz$

$$f''_{xx}=2>0, f''_{yy}=2>0, f''_{zz}=2>0$$

f convex in each variable on $[0,1]X[0,1]X[0,1]$ –compact.

By Gireaux's theorem f –has a minimum and a maximum value in one of the points: $(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1)$

$$f(0,0,0)=0, f(1,0,0)=f(0,1,0)=f(0,0,1)=1,$$
$$f(1,1,0)=f(0,1,1)=f(1,0,1)=f(1,1,1)=2$$

$$min\Omega=0, max\Omega=2$$

1262. If $x\in\left(0,\frac{\pi}{2}\right)$ then:

$$2(\sin^2 x+\csc^2 x)^2+2(\cos^2 x+\sec^2 x)^2\geq 25$$

Proposed by Daniel Sitaru – Romania

Solution by Pham Duc Nam-Vietnam

$$x\in\left(0,\frac{\pi}{2}\right)$$

$$2\left(\sin^2(x)+\frac{1}{\sin^2(x)}\right)^2+2\left(\cos^2(x)+\frac{1}{\cos^2(x)}\right)^2\geq 25?$$

$$*\ a=\sin^2(x), b=\cos^2(x), a+b=1$$

$$\Leftrightarrow 2\left(a+\frac{1}{a}\right)^2+2\left(b+\frac{1}{b}\right)^2\geq 2\cdot\frac{1}{2}\left(a+\frac{1}{a}+b+\frac{1}{b}\right)^2$$

$$= \left(1+\frac{a+b}{ab}\right)^2 \ge \left(1+\frac{1}{\frac{(a+b)^2}{4}}\right)^2 = 25$$

Equality holds iff $a = b = \frac{1}{2} \Leftrightarrow x = \frac{\pi}{4}$

1263. If $1 + a + b \neq 0$, then :

$$a^2 + b^2 + \left(\frac{6-a-b-ab}{1+a+b}\right)^2 \ge 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2 + b^2 + \left(\frac{6-a-b-ab}{1+a+b}\right)^2 = |a|^2 + |b|^2 + \left|\frac{6-a-b-ab}{1+a+b}\right|^2$$

$$\ge \frac{1}{3}\left(|a| + |b| + \left|\frac{6-a-b-ab}{1+a+b}\right|\right)^2 \overset{?}{\ge} 3 \Leftrightarrow \boxed{|a| + |b| + \left|\frac{6-a-b-ab}{1+a+b}\right| \underset{(*)}{\overset{?}{\ge}} 3}$$

Now, triangle inequality $\Rightarrow \forall\, a, b \in \mathbb{C} \mid 1 + a + b \neq 0$, we have :

$$|a| + |b| + \left|\frac{6-a-b-ab}{1+a+b}\right| \ge \left|a + b + \frac{6-a-b-ab}{1+a+b}\right| \overset{?}{\ge} 3$$

$$\Leftrightarrow \left|\frac{a + b + (a+b)^2 + 6 - a - b - ab}{1+a+b}\right| \overset{?}{\ge} 3$$

$$\Leftrightarrow \boxed{|a^2 + b^2 + ab + 6| \underset{(**)}{\overset{?}{\ge}} 3|1 + a + b|}$$

$$\text{Now, } a^2 + b^2 + ab + 6 \ge \frac{3}{4}(a+b)^2 + 6 \ge 6 > 0\ \forall\, a, b \in \mathbb{R} \mid 1 + a + b \neq 0$$

$$\therefore |a^2 + b^2 + ab + 6| = a^2 + b^2 + ab + 6 \ge \frac{3}{4}(a+b)^2 + 6$$

$$\forall\, a, b \in \mathbb{R} \mid 1 + a + b \neq 0 \to (1)$$

Also, $\forall\, a, b \in \mathbb{R} \mid 1 + a + b \neq 0$, we have via triangle inequality,

$$3|1 + a + b| \le 3(1 + |a + b|) \to (2) \therefore (1), (2)$$

$\Rightarrow$ in order to prove $(**)\ \forall\, a, b \in \mathbb{R} \mid 1 + a + b \neq 0$, it suffices to prove :

$$\frac{3}{4}(a+b)^2 + 6 \ge 3(1 + |a + b|) \Leftrightarrow |a + b|^2 + 8 \overset{?}{\ge} 4 + 4|a + b|$$

$$\Leftrightarrow |a + b|^2 - 4|a + b| + 4 \overset{?}{\ge} 0 \Leftrightarrow (|a + b| - 2)^2 \overset{?}{\ge} 0 \to \text{true} \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true } \forall\, a,b \in \mathbb{R} \,\big|\, 1+a+b \neq 0 \therefore a^2+b^2+\left(\frac{6-a-b-ab}{1+a+b}\right)^2 \geq 3$$

$$\forall\, a,b \in \mathbb{R} \,\big|\, 1+a+b \neq 0, \text{"="} \text{ iff } a=b=1 \text{ (QED)}$$

1264. If $\cos a+\cos b+\cos c=0$, then :

$$\cos^2 a+\cos^2 b+\cos^2 c \leq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\cos^2 a+\cos^2 b+\cos^2 c \leq 2 \overset{\cos a+\cos b+\cos c=0}{\Leftrightarrow}$$

$$\cos^2 a+\cos^2 b+(-\cos a-\cos b)^2 \leq 2 \Leftrightarrow 2(\cos^2 a+\cos^2 b)+2\cos a\cos b \leq 2$$

$$\Leftrightarrow \cos^2 a+\cos^2 b+\cos a\cos b \overset{(*)}{\leq} 1$$

Case 1: $\cos a, \cos b \geq 0$ and then : LHS of $(*) = (\cos a+\cos b)^2-\cos a\cos b$

$$\overset{\cos a+\cos b+\cos c=0}{=} (-\cos c)^2-\cos a\cos b \leq \cos^2 c \;(\because \cos a\cos b \geq 0) \leq 1$$

$$\Rightarrow (*) \text{ is true}$$

Case 2: $\cos a, \cos b \leq 0$ and then : LHS of $(*) = (\cos a+\cos b)^2-\cos a\cos b$

$$\overset{\cos a+\cos b+\cos c=0}{=} (-\cos c)^2-\cos a\cos b \leq \cos^2 c \;(\because \cos a\cos b \geq 0) \leq 1$$

$$\Rightarrow (*) \text{ is true}$$

Case 3i: $\cos a \geq 0, \cos b \leq 0$; $\cos c \geq 0$; LHS of $(*) = \cos^2 b+\cos a(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=} \cos^2 b-\cos a\cos c \leq \cos^2 b \;(\because \cos a\cos c \geq 0) \leq 1$$

$$\Rightarrow (*) \text{ is true}$$

Case3ii: $\cos a \geq 0, \cos b \leq 0$, $\cos c \leq 0$; LHS of $(*) = \cos^2 a+\cos b(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=} \cos^2 a-\cos b\cos c \leq \cos^2 a \;(\because \cos b\cos c \geq 0) \leq 1$$

$$\Rightarrow (*) \text{ is true}$$

Case 4i: $\cos a \leq 0, \cos b \geq 0$; $\cos c \geq 0$; LHS of $(*) = \cos^2 a+\cos b(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=} \cos^2 a-\cos b\cos c \leq \cos^2 a \;(\because \cos b\cos c \geq 0) \leq 1$$

$$\Rightarrow (*) \text{ is true}$$

Case4ii: $\cos a \leq 0, \cos b \geq 0$; $\cos c \leq 0$; LHS of $(*) = \cos^2 b+\cos a(\cos a+\cos b)$

$$\overset{\cos a+\cos b+\cos c=0}{=} \cos^2 b-\cos a\cos c \leq \cos^2 b \;(\because \cos a\cos c \geq 0) \leq 1$$

$$\Rightarrow (*) \text{ is true}$$

$$\therefore \text{combining } \textit{all cases}, (*) \text{ is true } \forall\, a,b \in \mathbb{R} \therefore \cos a+\cos b+\cos c=0$$

$$\Rightarrow \cos^2 a+\cos^2 b+\cos^2 c \leq 2 \;\forall\, a,b,c \in \mathbb{R} \text{ (QED)}$$

1265. If $a+b\neq 0$, then :

$$|a|+|b|+\left|\frac{3-ab}{a+b}\right|\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via triangle inequality}, \forall\, a,b\in\mathbb{C}\,\big|\, a+b\neq 0, \text{we have :}$$

$$|a|+|b|+\left|\frac{3-ab}{a+b}\right|\geq\left|a+b+\frac{3-ab}{a+b}\right|\overset{?}{\geq}3\Leftrightarrow\left|(a+b)^2+3-ab\right|\overset{?}{\geq}3|a+b|$$

$$\Leftrightarrow\left|a^2+b^2+ab+3\right|\underset{(*)}{\overset{?}{\geq}}3|a+b|$$

$$\text{Now}, a^2+b^2+ab+3\geq\frac{3}{4}(a+b)^2+3\geq 3>0\ \forall\, a,b\in\mathbb{R}\,\big|\,a+b\neq 0$$

$$\therefore\left|a^2+b^2+ab+3\right|=a^2+b^2+ab+3\geq\frac{3}{4}(a+b)^2+3\overset{?}{\geq}3|a+b|$$

$$\Leftrightarrow |a+b|^2+4\overset{?}{\geq}4|a+b|\Leftrightarrow(|a+b|-2)^2\overset{?}{\geq}0\to\text{true}\Rightarrow(*)\text{ is true}$$

$$\forall\, a,b\in\mathbb{R}\,\big|\,a+b\neq 0\ \therefore |a|+|b|+\left|\frac{3-ab}{a+b}\right|\geq 3\ \forall\, a,b\in\mathbb{R}\,\big|\,a+b\neq 0,$$

$$''=''\ \text{iff}\ (a=b=1)\text{ or }(a=b=-1)\ (\text{QED})$$

1266. If $x+1\neq 0$, then :

$$x^{2024}+\left(\frac{x-3}{x+1}\right)^{2024}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{For } \alpha=-1 \text{ and } \forall\, n\geq 1, (1+\alpha)^n-1-n\alpha=n-1\geq 0 \text{ and}$$
$$\forall\,\alpha>-1 \text{ and } \forall\, n\geq 1, (1+\alpha)^n\geq 1+n\alpha$$
$$\therefore\ \forall\,\alpha\geq 1 \text{ and } \forall\, n\geq 1, (1+\alpha)^n\geq 1+n\alpha\to(1)$$

$$\text{Now}, \forall\, x\in\mathbb{R}-\{-1\}, x^{2024}+\left(\frac{x-3}{x+1}\right)^{2024}$$

$$=\left(1+(x^2-1)\right)^{1012}+\left(1+\left(\left(\frac{x-3}{x+1}\right)^2-1\right)\right)^{2024}$$

$$\overset{\text{via }(1)}{\geq}1+1012(x^2-1)+1+1012\left(\left(\frac{x-3}{x+1}\right)^2-1\right)\overset{?}{\geq}2$$

$$\Leftrightarrow 1012(x^2-1)+1012\left(\left(\frac{x-3}{x+1}\right)^2-1\right)\overset{?}{\geq}0\Leftrightarrow x^2+\left(\frac{x-3}{x+1}\right)^2\overset{?}{\geq}2$$

$$\Leftrightarrow x^2(x+1)^2+(x-3)^2 \overset{?}{\geq} 2(x+1)^2 \Leftrightarrow x^4+2x^3-10x+7 \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-1)^2(x^2+4x+7) \overset{?}{\geq} 0 \Leftrightarrow (x-1)^2\left((x^2+4x+4)+3\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow (x-1)^2((x+2)^2+3) \overset{?}{\geq} 0 \to \text{true} \therefore x^{2024}+\left(\frac{x-3}{x+1}\right)^{2024} \geq 2$$

$$\forall\, x \in \mathbb{R}-\{-1\},''='' \text{ iff } x=1 \text{ (QED)}$$

Solution 2 by Eric Dimitrie Cismaru-Romania

Using Radon's Inequality, we have :

$$\frac{(x^2)^{1012}}{1^{1011}}+\frac{\left[\left(\frac{x-3}{x+1}\right)^2\right]^{1012}}{1^{1011}} \geq \frac{\left[x^2+\left(\frac{x-3}{x+1}\right)^2\right]^{1012}}{2^{1011}} \geq 2 \Leftrightarrow x^2+\left(\frac{x-3}{x+1}\right)^2 \geq 2 \Leftrightarrow$$

$$\Leftrightarrow x^2+\left(\frac{x-3}{x+1}\right)^2 \geq 2 \Leftrightarrow x^2+\frac{x^2-6x+9}{x^2+2x+1} \geq 2 \Leftrightarrow$$

$$\Leftrightarrow x^4+2x^3+2x^2-6x+9 \geq 2x^2+4x+2 \Leftrightarrow$$

$$\Leftrightarrow x^4+2x^3+7 \geq 10x \Leftrightarrow x^4+2x^3-10x+7 = x^4+3x^3-x^3-7x-3x+7 =$$

$$= x^3(x-1)+3x(x-1)(x+1)-7(x-1) =$$

$$= (x-1)[x^3+3x^2+3x-7] = (x-1)[(x+1)^3-2^3] =$$

$$= (x-1)^2[(x+1)^2+2(x+1)+4] =$$

$$= (x-1)^2(x^2+4x+7) = (x-1)^2[(x+2)^2+3] \geq 0,$$

so our inequality is proven. Equality holds iff $x=1$ (If $(x+2)^2+3=0$, we would have $(x+2)^2=-3\geq 0$, impossible).

1267. If $x,y>0, k\in\mathbb{N}^*$ then:

$$\sum_{n=1}^{k}\left(\sum_{m=1}^{n}\left(\frac{m}{\sqrt{x}}+\frac{\sqrt{y}}{n^2}\right)\sqrt{\frac{x}{m^2}+\frac{n^4}{y}}\right) \geq \sqrt{2}k(k+1)$$

Proposed by Khaled Abd Imouti-Syria

Solution by Daniel Sitaru-Romania

$$\sum_{n=1}^{k}\left(\sum_{m=1}^{n}\left(\frac{m}{\sqrt{x}}+\frac{\sqrt{y}}{n^2}\right)\sqrt{\frac{x}{m^2}+\frac{n^4}{y}}\right) \overset{AM-GM}{\geq} \sum_{n=1}^{k}\left(\sum_{m=1}^{n}\left(\frac{m}{\sqrt{x}}+\frac{\sqrt{y}}{n^2}\right)\sqrt{2\sqrt{\frac{x}{m^2}\cdot\frac{n^4}{y}}}\right) =$$

$$=\sum_{n=1}^{k}\left(\sum_{m=1}^{n}\left(\frac{m}{\sqrt{x}}+\frac{\sqrt{y}}{n^2}\right)\sqrt{2}\cdot\sqrt{\frac{\sqrt{x}}{m}\cdot\frac{n^2}{\sqrt{y}}}\right)\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{2}\sum_{n=1}^{k}\left(\sum_{m=1}^{n}\sqrt{\frac{m}{\sqrt{x}}\cdot\frac{\sqrt{y}}{n^2}\cdot\frac{\sqrt{x}}{m}\cdot\frac{n^2}{\sqrt{y}}}\right)=2\sqrt{2}\sum_{n=1}^{k}\left(\sum_{m=1}^{n}1\right)=$$

$$=2\sqrt{2}\sum_{n=1}^{k}n=2\sqrt{2}\cdot\frac{k(k+1)}{2}=\sqrt{2}k(k+1)$$

1268. If $a,\mathrm{b},\mathrm{c}>0$, *then* :

$$\frac{1}{a}+\frac{a}{\mathrm{b}}+a\mathrm{b}^2\geq\sqrt{3(1+a^2+\mathrm{b}^2)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{1}{a}+\frac{a}{\mathrm{b}}+a\mathrm{b}^2\geq\sqrt{3(1+a^2+\mathrm{b}^2)}\Leftrightarrow\left(\frac{\mathrm{b}+a^2+a^2\mathrm{b}^3}{a\mathrm{b}}\right)^2\geq 3(1+a^2+\mathrm{b}^2)$$

$$\Leftrightarrow a^4\mathrm{b}^6+2a^4\mathrm{b}^3+a^4+2a^2\mathrm{b}+\mathrm{b}^2\geq 3a^4\mathrm{b}^2+a^2\mathrm{b}^4+3a^2\mathrm{b}^2\to(1)$$

$$\text{Now, } a^4\mathrm{b}^3+a^4\mathrm{b}^3+a^4\overset{A-G}{\geq}3a^4\mathrm{b}^2\Rightarrow 2a^4\mathrm{b}^3+a^4\geq 3a^4\mathrm{b}^2\to(\mathrm{i})\therefore(\mathrm{i})\Rightarrow$$

$$\text{in order to prove (1), it suffices to prove : } a^4\mathrm{b}^5+2a^2+\mathrm{b}\geq a^2\mathrm{b}^3+3a^2\mathrm{b}$$

$$\Leftrightarrow a^4\mathrm{b}^5+\mathrm{b}+2a^2+a^2\mathrm{b}^3\geq 2a^2\mathrm{b}^3+3a^2\mathrm{b}\to(2)$$

$$\text{Now, } a^4\mathrm{b}^5+\mathrm{b}\overset{A-G}{\geq}2a^2\mathrm{b}^3\to(\mathrm{ii})\therefore(\mathrm{ii})\Rightarrow\text{to prove (2), it suffices to prove :}$$

$$2a^2+a^2\mathrm{b}^3\geq 3a^2\mathrm{b}\Leftrightarrow\mathrm{b}^3-3\mathrm{b}+2\geq 0\Leftrightarrow(\mathrm{b}+2)(\mathrm{b}-1)^2\geq 0\to\text{true}\because\mathrm{b}>0$$

$$\Rightarrow(2)\Rightarrow(1)\text{ is true}\therefore\frac{1}{a}+\frac{a}{\mathrm{b}}+a\mathrm{b}^2\geq\sqrt{3(1+a^2+\mathrm{b}^2)}\ \forall\, a,\mathrm{b},\mathrm{c}>0,$$

$$''='' \text{ iff } a=\mathrm{b}=\mathrm{c}=1\ (\text{QED})$$

1269. Prove that:

$$\sum_{k=1}^{n}e^{k\cdot k!}>\frac{n}{\sqrt[n]{e}}\cdot e^{\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n}},\qquad n\geq 2, n\in\mathbb{N}$$

Proposed by Khaled Abd Imouti-Syria

Solution by Daniel Sitaru-Romania

Lemma 1:

$$1\cdot 1!+2\cdot 2!+\cdots+n\cdot n!=(n+1)!-1,\qquad n\in\mathbb{N}^*$$

Proof: For $n = 1 \to 1 \cdot 1! = (1+1)! - 1$.True.

$$P(n): \sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$$

→suppose true

$$P(n+1): \sum_{k=1}^{n+1} k \cdot k! = (n+2)! - 1$$

→to prove

$$\sum_{k=1}^{n+1} k \cdot k! = \sum_{k=1}^{n} k \cdot k! + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)!$$
$$= (n+1)!\,(n+1+1) - 1 = (n+2)! - 1$$
$$P(n) \to P(n+1)$$

Lemma 2:

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n, \qquad n \in \mathbb{N}$$

Proof:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot a^{n-k} \cdot b^k$$

For $a = b = 1$:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

Lemma 3:

$$(n+1)! \geq 2^n, \qquad n \in \mathbb{N}$$

Proof: For $n = 0 \to 1! > 2^0$.True.

$$P(n): (n+1)! \geq 2^n$$

→suppose true

$$P(n+1): (n+2)! \geq 2^{n+1}$$

→to prove

$$(n+2)! = (n+1)!\,(n+2) \geq 2^n(n+2) \geq 2^{n+1} \Leftrightarrow n+2 \geq 2$$
$$P(n) \to P(n+1)$$

Back to the problem:

$$\sum_{k=1}^{n} e^{k\cdot k!} \overset{AM-GM}{\geq} n\sqrt[n]{\prod_{k=1}^{n} e^{k\cdot k!}} = n\sqrt[n]{e^{\sum_{k=1}^{n} k\cdot k!}} \overset{Lemma\ 1}{=} n\sqrt[n]{e^{(n+1)!-1}} =$$

$$= \frac{n}{\sqrt[n]{e}}\cdot\sqrt[n]{e^{(n+1)!}} > \frac{n}{\sqrt[n]{e}}\cdot e^{\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n}} \Leftrightarrow$$

$$\Leftrightarrow \sqrt[n]{e^{(n+1)!}} > e^{\frac{\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}}{n}} \overset{Lemma\ 2}{\Leftrightarrow} \sqrt[n]{e^{(n+1)!}} > e^{\frac{2^n}{n}} \Leftrightarrow$$

$$\Leftrightarrow \sqrt[n]{e^{(n+1)!}} > \sqrt[n]{e^{2^n}} \Leftrightarrow (n+1)! > 2^n.$$ True by Lemma 3.

1270. If $x, y, z > 0$ and $x^2 + y^2 + z^2 \leq 3y$, then prove that :

$$\frac{1}{(x+1)^2} + \frac{4}{(y+2)^2} + \frac{8}{(z+3)^2} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{z^2}{1} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} \overset{\text{Bergstrom}}{\geq} \frac{(z+1+1+1)^2}{4} \Rightarrow (z+3)^2 \leq 4(z^2+3)$$

$$\therefore \frac{1}{(x+1)^2} + \frac{8}{(z+3)^2} \geq \frac{1}{2(x^2+1)} + \frac{8}{4(z^2+3)}$$

$$= \frac{1}{2(x^2+1)} + \frac{1}{z^2+3} + \frac{1}{z^2+3} \overset{\text{Bergstrom}}{\geq} \frac{9}{2(x^2+z^2)+8}$$

$$\overset{x^2+y^2+z^2\leq 3y}{\geq} \frac{9}{2(3y-y^2)+8} \Rightarrow \frac{1}{(x+1)^2} + \frac{4}{(y+2)^2} + \frac{8}{(z+3)^2} - 1$$

$$\geq \frac{9}{6y-2y^2+8} + \frac{4}{(y+2)^2} - 1$$

$$= \frac{9(y+2)^2 + 4(6y-2y^2+8) - (6y-2y^2+8)(y+2)^2}{(6y-2y^2+8)(y+2)^2} \overset{?}{\geq} 0$$

$$\Leftrightarrow 9(y+2)^2 + 4(6y-2y^2+8) - (6y-2y^2+8)(y+2)^2 \overset{?}{\geq} 0$$

$$\left(\because 6y-2y^2+8 = 2(3y-y^2)+8 \overset{x^2+y^2+z^2\leq 3y}{\geq} 2x^2+2z^2+8 > 0\right)$$

$$\Leftrightarrow 2y^4 + 2y^3 - 23y^2 + 4y + 36 \overset{?}{\geq} 0 \Leftrightarrow (y-2)^2(2y^2+10y+9) \overset{?}{\geq} 0 \to \text{true}$$

$$\Rightarrow \frac{1}{(x+1)^2} + \frac{4}{(y+2)^2} + \frac{8}{(z+3)^2} \geq 1\ \forall x,y,z > 0 \mid x^2+y^2+z^2 \leq 3y,$$

$$'' ='' \text{ iff } x = z = 1 \text{ and } y = 2 \text{ (QED)}$$

1271. If $x, y, z > 0$ and $xyz = 1$, then prove that :

$$\frac{(\sqrt{2}x)^2}{(1+xz)(1+xy)} + \frac{(\sqrt{2}y)^2}{(1+yz)(1+xy)} + \frac{(\sqrt{2}z)^2}{(1+xz)(1+yz)} \ge \frac{3}{2}$$

Proposed by Lamiye Quliyeva-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x^2}{(1+xz)(1+xy)} \overset{xyz=1}{=} \frac{x^2}{(xyz+xz)(xyz+xy)} = \frac{1}{yz(y+1)(z+1)}$$

$$\overset{xyz=1}{=} \frac{x(x+1)}{(x+1)(y+1)(z+1)} \text{ and analogs}$$

$$\therefore \frac{x^2}{(1+xz)(1+xy)} + \frac{y^2}{(1+yz)(1+xy)} + \frac{z^2}{(1+xz)(1+yz)}$$

$$= \sum_{cyc} \frac{x(x+1)}{(x+1)(y+1)(z+1)} \overset{?}{\ge} \frac{3}{4} \Leftrightarrow 4\sum_{cyc} x^2 + \sum_{cyc} x \underbrace{\overset{?}{\ge}}_{(*)} 3xyz + 3 + 3\sum_{cyc} xy$$

$$\text{Now, } 4\sum_{cyc} x^2 + \sum_{cyc} x \ge \sum_{cyc} x^2 + 3\sum_{cyc} xy + \sum_{cyc} x \overset{A-G}{\ge}$$

$$3\sqrt[3]{x^2y^2z^2} + 3\sum_{cyc} xy + 3\sqrt[3]{xyz} \overset{xyz=1}{=} 3 + 3 + 3\sum_{cyc} xy \overset{xyz=1}{=} 3xyz + 3 + 3\sum_{cyc} xy$$

$$\Rightarrow (*) \text{ is true } \therefore \frac{x^2}{(1+xz)(1+xy)} + \frac{y^2}{(1+yz)(1+xy)} + \frac{z^2}{(1+xz)(1+yz)} \ge \frac{3}{4}$$

$$\Rightarrow \frac{(\sqrt{2}x)^2}{(1+xz)(1+xy)} + \frac{(\sqrt{2}y)^2}{(1+yz)(1+xy)} + \frac{(\sqrt{2}z)^2}{(1+xz)(1+yz)} \ge \frac{3}{2}$$

$$\forall x, y, z > 0 \mid xyz = 1, \text{"="} \text{ iff } x = y = z = 1$$

Solution 2 by Pham Duc Nam-Vietnam

$$\frac{2x^2}{(1+xz)(1+xy)} + \frac{2y^2}{(1+yz)(1+xy)} + \frac{2z^2}{(1+xz)(1+yz)} \ge \frac{3}{2}$$

$$*x + y + z \ge 3\sqrt[3]{xyz} = 3. \text{ And:}$$

$$(x+y+z)^2 \ge 3(xy+yz+xz) \Longrightarrow 2(xy+yz+xz) \le \frac{2(x+y+z)^2}{3}$$

$$*(1+xz)(1+xy) = xxyz + xy + xz + 1 = x + xy + xz + xyz = x(1+y)(1+z) \Rightarrow$$

$$(1+yz)(1+xy) = y(1+x)(1+z), (1+xz)(1+yz) = z(1+x)(1+y) \Rightarrow$$

$$\frac{2x^2}{(1+xz)(1+xy)}+\frac{2y^2}{(1+xy)(1+yz)}+\frac{2z^2}{(1+xz)(1+yz)}\overset{Bergstrom's}{\geq}$$

$$\geq \frac{2(x+y+z)^2}{x(1+y)(1+z)+y(1+x)(1+z)+z(1+x)(1+y)}=$$

$$=\frac{2(x+y+z)^2}{x+y+z+3xyz+2(xy+yz+xz)}\geq \frac{2(x+y+z)^2}{\frac{2}{3}(x+y+z)^2+(x+y+z)+3}$$

$$\because f(t)=\frac{2t^2}{\frac{2}{3}t^2+t+3}, t\geq 3, f''=\frac{18t(t+6)}{(2t^2+3t+9)^2}>0 \forall t\geq 3\Rightarrow$$

$$\frac{2x^2}{(1+xz)(1+xy)}+\frac{2y^2}{(1+xy)(1+yz)}+\frac{2z^2}{(1+xz)(1+yz)}\geq \frac{2}{3}$$

Equality holds iff : x=y=z=1

1272. **If $a, b\in\mathbb{R}$ and $a^2+b^2\leq a+b$, then prove that :**

$$a(a+1)+b(b+1)\leq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a+b\geq a^2+b^2\geq \frac{(a+b)^2}{2}\Rightarrow 2\geq a+b\ (\because a+b\geq a^2+b^2\geq 0)$$

$$\Rightarrow 4\geq 2a+2b\overset{a+b\,\geq\, a^2+b^2}{\geq} a+b+a^2+b^2\Rightarrow 4\geq a(a+1)+b(b+1),$$

$$''=''\ \text{iff } a=b=1\ (\text{QED})$$

1273. **If $a, b\in\mathbb{R}$ and $a^2+b^2+a+b=ab$, then prove that :**

$$a^3+b^3\geq -16$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=a+b \text{ and then}: a^2+b^2+a+b=ab\Rightarrow -x=a^2+b^2-ab$$

$$\geq \frac{(a+b)^2}{4}=\frac{x^2}{4}\Rightarrow x^2+4x\leq 0\Rightarrow x(x+4)\leq 0\Rightarrow -4\leq x\leq 0\rightarrow (1)$$

$$\text{Now, } a^3+b^3=(a+b)^3-3ab(a+b)\overset{a^2+b^2+a+b\,=\,ab}{=}x^3-3x(a^2+b^2+a+b)$$

$$\geq x^3-3x\left(\frac{(a+b)^2}{2}+a+b\right)\left(\because -3x\geq 0 \text{ and } a^2+b^2\geq\frac{(a+b)^2}{2}\right)\overset{?}{\geq}-16$$

$$\Leftrightarrow x^3 - 3x\left(\frac{x^2}{2} + x\right) \overset{?}{\geq} -16 \Leftrightarrow x^3 + 6x^2 - 32 \overset{?}{\leq} 0 \Leftrightarrow (x+4)^2(x-2) \overset{?}{\leq} 0 \to \text{true}$$

$$\because x \overset{\text{via (1)}}{\leq} 0 < 2 \Rightarrow x - 2 < 0, " = " \text{ iff } x = a + b = -4$$

$$\text{and for } a = b \Rightarrow \text{iff } a = b = -2 \therefore a^3 + b^3 \geq -16$$

$$\forall\, a, b \in \mathbb{R} \,\middle|\, a^2 + b^2 + a + b = ab, " = " \text{ iff } a = b = -2 \text{ (QED)}$$

1274. If $a, b \in \mathbb{R}$ and $a^5 + b^5 = a^2 + b^2$, then prove that :

$$a^3 + b^3 \leq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^4 - a^3b + a^2b^2 - ab^3 + b^4 = a^3(a-b) - b^3(a-b) + a^2b^2 =$$

$$(a-b)^2(a^2 + b^2 + ab) + a^2b^2 = (a-b)^2\left(\frac{3}{4}(a+b)^2 + \frac{1}{4}(a-b)^2\right) + a^2b^2 \geq 0$$

$$\Rightarrow a^4 - a^3b + a^2b^2 - ab^3 + b^4 \geq 0 \Rightarrow (a^2 + b^2)^2 - a^2b^2 - ab(a^2 + b^2) \geq 0$$

$$\Rightarrow x^2 - y^2 - xy \overset{(1)}{\geq} 0 \; (x = a^2 + b^2, y = ab)$$

$$\text{Now, } a^5 + b^5 = a^2 + b^2 \Rightarrow (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) = a^2 + b^2$$

$$\Rightarrow (a+b)(x^2 - y^2 - xy) \overset{(*)}{=} x \text{ and (1) suggests 2 cases :}$$

$$\boxed{\text{Case (i)}}\; x^2 - y^2 - xy = 0 \text{ and then : via } (*), x = 0 \text{ and plugging } x = 0$$

$$\text{in } x^2 - y^2 - xy = 0, \text{ we get : } y = 0 \therefore a^2 + b^2 = ab = 0 \Rightarrow (a+b)(a^2 + b^2 - ab)$$

$$= 0 \Rightarrow a^3 + b^3 = 0 \therefore a^3 + b^3 < 2$$

$$\boxed{\text{Case (ii)}}\; x^2 - y^2 - xy > 0 \text{ and then : via } (*), a + b = \frac{x}{x^2 - y^2 - xy} \Rightarrow$$

$$a^3 + b^3 - 2 = (a+b)(a^2 + b^2 - ab) - 2 = \frac{x(x-y)}{x^2 - y^2 - xy} - 2 = -\frac{x^2 - xy - 2y^2}{x^2 - y^2 - xy}$$

$$= -\frac{(x - 2y)(x + y)}{x^2 - y^2 - xy} = -\frac{(a^2 + b^2 - 2ab)(a^2 + b^2 + ab)}{x^2 - y^2 - xy} =$$

$$-\frac{(a-b)^2\left(\frac{3}{4}(a+b)^2 + \frac{1}{4}(a-b)^2\right)}{x^2 - y^2 - xy} \leq 0 \therefore a^3 + b^3 \leq 2 \text{ and combining both cases,}$$

$$a^3 + b^3 \leq 2 \; \forall\, a, b \in \mathbb{R} \,\middle|\, a^5 + b^5 = a^2 + b^2, " = " \text{ iff } a = b = 1 \text{ (QED)}$$

1275. If $a, b, c > 0, \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq 3$ then:

$$\frac{1}{(a+1)^2} + \frac{1}{(b+1)^2} + \frac{1}{(c+1)^2} \leq \frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{(a+1)^2}+\frac{1}{(b+1)^2}+\frac{1}{(c+1)^2}\overset{AM-GM}{\leq}\sum\frac{1}{\left(2\sqrt{a}\right)^2}=$$

$$=\frac{1}{4}\sum\frac{1}{a}=\frac{1}{4}\sum\sqrt{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2}\leq$$

$$\leq\frac{1}{4}\sum\sqrt{\left(3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\right)}\leq\frac{1}{4}\sqrt{9}=\frac{3}{4}\left(since,\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\leq 3\right)$$

1276. If $a,b,c>0,\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\leq 3$ then:

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\leq\frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\overset{AM-GM}{\leq}\sum\frac{1}{4ab}=\frac{1}{4}\sum\frac{1}{ab}\leq$$

$$\leq\frac{1}{4}\sum\frac{1}{a^2}\leq\frac{3}{4}\left(since\ \sum\frac{1}{a^2}\leq 3\right)$$

1277. If $a,b\in\mathbb{R}$ and $ab(a^3+b^3)=2$, then prove that :

$$a^2+b^2\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+b^2\geq 2\Leftrightarrow(a^2+b^2)^5\geq 32\overset{4=\left(ab(a^3+b^3)\right)^2}{=}8\left(ab(a^3+b^3)\right)^2$$

$$=8a^2b^2(a+b)^2(a^2+b^2-ab)^2=8a^2b^2(a^2+b^2+2ab)(a^2+b^2-ab)^2$$

$$\Leftrightarrow x^5\geq 8y^2(x+2y)(x-y)^2\ (x=a^2+b^2,y=ab)$$

$$\Leftrightarrow x^5-8x^3y^2+24xy^4-16y^5\geq 0$$

$$\Leftrightarrow x^5-2x^4y+2x^4y-4x^3y^2-4x^3y^2+8x^2y^3-8x^2y^3$$

$$+16xy^4+8xy^4-16y^5\geq 0\Leftrightarrow$$

$$x^4(x-2y)+2x^3y(x-2y)-4x^2y^2(x-2y)-8xy^3(x-2y)+8y^4(x-2y)\geq 0$$

$$\Leftrightarrow(x-2y)(x^4+2x^3y-4x^2y^2-8xy^3+8y^4)\geq 0$$

$$\Leftrightarrow (x-2y)(x(x-2y)(x^2+4xy+4y^2)+8y^4)\geq 0$$

$$\Leftrightarrow (a^2+b^2-2ab)\left((a^2+b^2)(a^2+b^2-2ab)b^2(a^2+b^2+2ab)^2+8a^4b^4\right)\geq 0$$

$$\Leftrightarrow (a-b)^2\left((a^2+b^2)(a-b)^2(a+b)^4+8a^4b^4\right)\geq 0\to \text{true}\ \forall\, a,b\in\mathbb{R}$$

$$\therefore a^2+b^2\geq 2\ \forall\, a,b\in\mathbb{R}\,\Big|\,ab(a^3+b^3)=2,''=''\ \text{ iff } a=b=1\ (\text{QED})$$

1278. If $a,b,c>0$ and $a+b+c=6$, then prove that :

$$\frac{a}{\sqrt{b^3+1}}+\frac{b}{\sqrt{c^3+1}}+\frac{c}{\sqrt{a^3+1}}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We shall prove that : $\frac{1}{\sqrt{x^3+1}}\overset{(*)}{\geq}\frac{7-2x}{9}\ \forall\, x\in(0,6)$ and we note that if $x\in\left[\frac{7}{2},6\right)$, then : RHS of $(*)\leq 0<$ LHS of $(*)\Rightarrow(*)$ is true and so, we now focus on : $x\in\left(0,\frac{7}{2}\right)$ and then : $\frac{7-2x}{9}>0\ \therefore(*)\Leftrightarrow\frac{1}{x^3+1}\geq\frac{(7-2x)^2}{81}$

$$\Leftrightarrow 4x^5-28x^4+49x^3+4x^2-28x-32\leq 0\Leftrightarrow (x-2)^2(4x^3-12x^2-15x-8)$$

$$\leq 0\Leftrightarrow (x-2)^2\left((x-4)(2x+1)^2-4\right)\leq 0\to\text{true}\ \because x<\frac{7}{2}<4$$

$\Rightarrow (x-4)(2x+1)^2-4<0\ \therefore(*)$ is true is true and combining both cases, $(*)$ is true $\forall\, x\in(0,6)$ and via $(*)$, $\frac{a}{\sqrt{b^3+1}}+\frac{b}{\sqrt{c^3+1}}+\frac{c}{\sqrt{a^3+1}}\geq\sum_{cyc}\frac{a(7-2b)}{9}$

$$=\frac{7}{9}\sum_{cyc}a-\frac{2}{9}\sum_{cyc}ab\geq\frac{7}{9}\sum_{cyc}a-\frac{2}{27}\left(\sum_{cyc}a\right)^2\overset{a+b+c=6}{=}\frac{7}{9}.6-\frac{2}{27}.36=2$$

$$\therefore\frac{a}{\sqrt{b^3+1}}+\frac{b}{\sqrt{c^3+1}}+\frac{c}{\sqrt{a^3+1}}\geq 2$$

$$\forall\, a,b,c>0\,\Big|\,a+b+c=6,''=''\ \text{ iff } a=b=c=2\ (\text{QED})$$

1279. If $a,b,c>0$ and $a+b+c=3$, then prove that :

$$a^2+b^2+c^2+\frac{ab+bc+ca}{a^2b+b^2c+c^2a}\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x,y,z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \text{and } \sum_{cyc} a^2b^2$$

$$= \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \to (5)$$

$$\text{Now, } \frac{\left(9\sum_{cyc} a^2 - 2\left(\sum_{cyc} a\right)^2\right)^2}{9\left(\sum_{cyc} a\right)^2} \overset{?}{\geq} \frac{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}{\left(\sum_{cyc} ab\right)^2} \overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow}$$

$$\frac{(9(s^2 - 8Rr - 2r^2) - 2s^2)^2}{9s^2} \overset{?}{\geq} \frac{r^2(s^2 - 8Rr - 2r^2)((4R+r)^2 - 2s^2)}{(4Rr + r^2)^2}$$

$$\Leftrightarrow 9s^6 + (320R^2 + 88Rr + 2r^2)s^4 - r(7488R^3 + 5616R^2r + 1404Rr^2 + 117r^3)s^2$$

$$+162r^2(4R+r)^4 \underbrace{\overset{?}{\geq}}_{(*)} 0 \text{ and } \because 9(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to}$$

$$\text{prove } (*), \text{it suffices to prove : LHS of } (*) \geq 9(s^2 - 16Rr + 5r^2)^3$$

$$\Leftrightarrow (320R^2 + 520Rr - 133r^2)s^4 - r(7488R^3 + 12528R^2r - 2916Rr^2 + 792r^3)s^2$$

$$+r^2(41472R^4 + 78336R^3r - 19008R^2r^2 + 13392Rr^3 - 963r^4) \overset{(**)}{\geq} 0 \text{ and}$$

$$\because (320R^2 + 520Rr - 133r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove } (**), \text{it suffices to prove : LHS of } (**) \geq$$

$$(320R^2 + 520Rr - 133r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (1376R^3 + 456R^2r - 3270Rr^2 + 269r^3)s^2$$

$$\overset{(***)}{\geq} r(20224R^4 + 1792R^3r - 45120R^2r^2 + 10444Rr^3 - 1181r^4) \text{ and } \because$$

$$1376R^3 + 456R^2r - 3270Rr^2 + 269r^3$$

$$= (R-2r)(1376R^2+3208Rr+3146r^2)+6561r^3 \overset{\text{Euler}}{\geq} 6561r^3 > 0$$

$$\therefore \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq} (1376R^3+456R^2r-3270Rr^2+269r^3)(16Rr-5r^2)$$

$$\overset{?}{\geq} r(20224R^4+1792R^3r-45120R^2r^2+10444Rr^3-1181r^4)$$

$$\Leftrightarrow 896t^4-688t^3-4740t^2+5105t-82 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)(896t^2+2896t+3260)+6561\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{9\sum_{cyc}a^2-2\left(\sum_{cyc}a\right)^2}{3\sum_{cyc}a} \geq \frac{\sqrt{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)}}{\sum_{cyc}ab}$$

$$\overset{\text{Reverse CBS}}{\geq} \frac{\sum_{cyc}a^2b}{\sum_{cyc}ab} \overset{a+b+c=3}{\Rightarrow} \sum_{cyc}a^2-2 \geq \frac{\sum_{cyc}a^2b}{\sum_{cyc}ab} \Rightarrow \sum_{cyc}a^2+\frac{\sum_{cyc}ab}{\sum_{cyc}a^2b}$$

$$\geq 2+\frac{\sum_{cyc}a^2b}{\sum_{cyc}ab}+\frac{\sum_{cyc}ab}{\sum_{cyc}a^2b} \overset{A-G}{\geq} 2+2=4 \therefore a^2+b^2+c^2+\frac{ab+bc+ca}{a^2b+b^2c+c^2a} \geq 4$$

$$\forall a,b,c>0 \mid a+b+c=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

1280. If $a, b \in \mathbb{R}$ and $a^2+b^2+ab=3(a+b)$, then prove that :

$$a^3+b^3 \leq 27$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

If $a=0$, then : $b^2=3b \Rightarrow b=0$ or $b=3$; similarly, if $b=0$, then : $a^2=3a \Rightarrow a=0$ or $a=3$. So, if $(a,b)=(0,0)$ or $(0,3)$ or $(3,0)$, we see that : $a^3+b^3 \leq 27$ is true, $''=''$ iff $(a=3, b=0)$ or $(a=0, b=3)$ and we now shift our attention to : $a \neq 0, b \neq 0 \ldots$ then : $a^2+b^2+ab=$

$$b^2(t^2+t+1)\ \left(t=\frac{a}{b}\right) = b^2\left(\left(t+\frac{1}{2}\right)^2+\frac{3}{4}\right) \geq \frac{3b^2}{4} > 0\ (\because b \neq 0)$$

$$\Rightarrow 3(a+b)=a^2+b^2+ab>0 \therefore a^3+b^3 \leq 27 \overset{\because a^2+b^2+ab=3(a+b)}{\Leftrightarrow}$$

$$a^3+b^3 \leq \left(\frac{a^2+b^2+ab}{a+b}\right)^3 \Leftrightarrow (a^2+b^2+ab)^3 \geq (a^3+b^3)(a+b)^3$$

$$\left(\because a^2+b^2+ab>0 \text{ and } a+b>0\right)$$

$$\Leftrightarrow (a^2+b^2+ab)^3 \geq (a^2+b^2-ab)(a+b)^4 = (a^2+b^2-ab)(a^2+b^2+2ab)^2$$

$$\Leftrightarrow (x+y)^3 \geq (x-y)(x+2y)^2 \Leftrightarrow y^2(3x+5y) \geq 0$$

$$\Leftrightarrow a^2b^2(3a^2+3b^2+5ab) \geq 0 \Leftrightarrow a^2b^4(3t^2+5t+3) \geq 0 \left(x=a^2+b^2 \text{ and } y=ab\right)$$

$\to$ true (strict inequality) $\because a^2b^4>0$ $(\because a, b \neq 0)$ and discriminant of

$$3t^2 + 5t + 3 - 25 - 36 < 0 \Rightarrow 3t^2 + 5t + 3 > 0 \therefore a^3 + b^3 < 27 \text{ for } a \neq 0, b \neq 0$$
$$\text{and combining } all \text{ cases}, a^3 + b^3 \leq 27 \,\forall\, a, b \in \mathbb{R} \,\big|\, a^2 + b^2 + ab = 3(a + b),$$
$$'' ='' \text{ iff } (a = 3, b = 0) \text{ or } (a = 0, b = 3) \text{ (QED)}$$

1281. If $a \geq b \geq c > 0$, then prove that :

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } b = c + x \ (x \geq 0) \text{ and } a = b + y \ (y \geq 0) = c + x + y \text{ and then :}$$

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0$$

$$\Leftrightarrow (c + x + y)^2(c + x)y + (c + x)^2cx + c^2(c + x + y)(-x - y) \geq 0$$

$$\Leftrightarrow c^2x^2 + c^2xy + c^2y^2 + cx^3 + 3cx^2y + 4cxy^2 + cy^3 + x^3y + 2x^2y^2 + xy^3 \geq 0$$

$$\rightarrow \text{true} \because c > 0 \text{ and } x, y \geq 0 \therefore a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0$$

$$\forall\, a \geq b \geq c > 0, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

1282. If $x, y, z > 0, x + y + z = 1$ and $\lambda \geq \frac{2}{27}$, then :

$$\sum_{cyc} \frac{\lambda - x^3}{x} \geq \frac{27\lambda - 1}{3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Assigning } y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a$$
$$= 2x > 0 \text{ and } c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b \Rightarrow a, b, c \text{ form}$$
$$\text{sides of } a \text{ triangle with semiperimeter, circumradius and inradius} = s, R, r \text{ (say)}$$
$$\text{yielding } 2\sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x \overset{(\bullet)}{=} s \Rightarrow x = s - a, y = s - b, z = s - c$$
$$\therefore xyz \overset{(\bullet\bullet)}{=} r^2s \text{ and}, \sum_{cyc} xy = \sum_{cyc} (s - a)(s - b) = 4Rr + r^2 \Rightarrow \sum_{cyc} xy \overset{(\bullet\bullet\bullet)}{=} 4Rr + r^2$$

$$\text{and also}, \sum_{cyc} x^2 = \left(\sum_{cyc} x\right)^2 - 2\sum_{cyc} xy \overset{\text{via } (\bullet) \text{ and } (\bullet\bullet\bullet)}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} x^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s^2 - 8Rr - 2r^2$$

$$\sum_{cyc} \frac{\lambda - x^3}{x} \ge \frac{27\lambda - 1}{3} \Leftrightarrow \lambda\left(\frac{\sum_{cyc} xy - 9xyz}{xyz}\right) \ge \sum_{cyc} x^2 - \frac{1}{3} \overset{x+y+z=1}{\Leftrightarrow}$$

$$\lambda\left(\frac{(\sum_{cyc} x)(\sum_{cyc} xy) - 9xyz}{xyz}\right) \boxed{\overset{(*)}{\ge}} \frac{\sum_{cyc} x^2}{(\sum_{cyc} x)^2} - \frac{1}{3}$$

$$\text{Now}, \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{A-G}{\ge} 9xyz \Rightarrow \frac{(\sum_{cyc} x)(\sum_{cyc} xy) - 9xyz}{xyz} \ge 0 \text{ and } \because \lambda \ge \frac{2}{27}$$

$$\therefore \text{LHS of } (*) \ge \frac{2}{27}\left(\frac{(\sum_{cyc} x)(\sum_{cyc} xy) - 9xyz}{xyz}\right) \overset{?}{\ge} \frac{\sum_{cyc} x^2}{(\sum_{cyc} x)^2} - \frac{1}{3}$$

$$\overset{\text{via } (\bullet),(\bullet\bullet),(\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet)}{\Leftrightarrow} \frac{2}{27} \cdot \frac{s(4Rr + r^2) - 9r^2 s}{r^2 s} \overset{?}{\ge} \frac{3(s^2 - 8Rr - 2r^2) - s^2}{3s^2}$$

$$\Leftrightarrow \frac{s^2(4R - 8r)}{9r} \boxed{\underset{(**)}{\overset{?}{\ge}}} s^2 - 12Rr - 3r^2$$

$$\text{Again}, s^2 - 12Rr - 3r^2 \underset{(i)}{\underbrace{\overset{\text{Gerretsen}}{\le}}} 4R^2 - 8Rr \text{ and}$$

$$\frac{s^2(4R - 8r)}{9r} \underset{(ii)}{\underbrace{\overset{\text{Gerretsen}}{\ge}}} \frac{(4R - 8r)(16R - 5r)}{9}$$

$$\therefore (i), (ii) \Rightarrow \text{in order to prove } (**), \text{it suffices to prove :}$$

$$\frac{(4R - 8r)(16R - 5r)}{9} \ge 4R^2 - 8Rr \Leftrightarrow 4(R - 2r)(7R - 5r) \ge 0 \to \text{true via Euler}$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{\lambda - x^3}{x} \ge \frac{27\lambda - 1}{3} \ \forall\, x, y, z > 0 \,|\, x + y + z = 1$$

$$\text{and } \lambda \ge \frac{2}{27}, \text{"="} \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

The given inequality can be rewritten as follows

$$\lambda\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 9\right) \ge x^2 + y^2 + z^2 - \frac{1}{3}.$$

Since $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge \frac{9}{x + y + z} = 9$, so it suffices to prove that

$$\frac{2}{27}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-9\right)\geq x^2+y^2+z^2-\frac{1}{3}\ \text{or}$$

$$\frac{2}{27}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)+2(xy+yz+zx)\geq\frac{4}{3}\ (1)$$

$$\blacksquare LHS_{(1)}\overset{AM-GM}{\geq}2\sqrt{\frac{4}{27}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)(xy+yz+zx)}=\frac{4}{3}\sqrt{\frac{(xy+yz+zx)^2}{3xyz}}$$

$$\geq\frac{4}{3}\sqrt{x+y+z}=\frac{4}{3}.$$

So the proof is complete. Equality holds iff $x=y=z=\frac{1}{3}$.

1283. If $a,b,c>0$ and $\lambda\geq\frac{1}{2}$ then:

$$\frac{a}{\sqrt{ab+\lambda b^2}}+\frac{b}{\sqrt{bc+\lambda c^2}}+\frac{c}{\sqrt{ca+\lambda a^2}}\geq\frac{3}{\sqrt{\lambda+1}}$$

Proposed by Marin Chirciu-Romania

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\sum_{cyc}\frac{a}{\sqrt{ab+\lambda b^2}}=2\sqrt{\lambda+1}.\sum_{cyc}\frac{a}{2\sqrt{b(\lambda+1).(a+\lambda b)}}\overset{AM-GM}{\geq}$$

$$\geq 2\sqrt{\lambda+1}.\sum_{cyc}\frac{a}{b(\lambda+1)+(a+\lambda b)}\geq$$

$$\overset{CBS}{\geq}2\sqrt{\lambda+1}.\frac{(a+b+c)^2}{\sum_{cyc}a[a+(1+2\lambda)b]}=2\sqrt{\lambda+1}.\frac{(a+b+c)^2}{\left(\sum_{cyc}a\right)^2+(2\lambda-1)\sum_{cyc}bc}\geq$$

$$\geq 2\sqrt{\lambda+1}.\frac{(a+b+c)^2}{\left(\sum_{cyc}a\right)^2+(2\lambda-1).\frac{\left(\sum_{cyc}a\right)^2}{3}}=\frac{3}{\sqrt{\lambda+1}}.$$

Equality holds iff $a=b=c$.

1284. If $a, \mathrm{b} \in \mathbb{R}$ and $a^3 + \mathrm{b}^3 + a^2 + \mathrm{b}^2 = 4$, then prove that :

$$a^4 + \mathrm{b}^4 \geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$a^3 + \mathrm{b}^3 + a^2 + \mathrm{b}^2 = 4 \Rightarrow (a+\mathrm{b})^3 - 3a\mathrm{b}(a+\mathrm{b}) + (a+\mathrm{b})^2 - 2a\mathrm{b} = 4$

$\Rightarrow x^3 - 3x\mathrm{y} + x^2 - 2\mathrm{y} = 4\ (x = a+\mathrm{b}, \mathrm{y} = a\mathrm{b}) \Rightarrow x^3 + x^2 - 4 = \mathrm{y}(3x+2)$

$$\Rightarrow \frac{x^3+x^2-4}{3x+2} \overset{(\bullet)}{=} \mathrm{y} \left(\begin{array}{c}\text{if } 3x+2=0, \text{then} : x = -\frac{2}{3}, \text{but } x^3+x^2-4 \neq 0 \text{ for} \\ x = -\frac{2}{3} \Rightarrow 3x+2 \neq 0\end{array}\right)$$

$$\leq \frac{x^2}{4} \Rightarrow \frac{4(x^3+x^2-4) - x^2(3x+2)}{4(3x+2)} \leq 0 \Rightarrow \frac{(3x+2)(x-2)\left((x+2)^2+4\right)}{(3x+2)^2} \leq 0$$

$$\Rightarrow \boxed{-\frac{2}{3} < x \leq 2} \to (1)$$

Now, $a^4 + \mathrm{b}^4 = \left(a^2+\mathrm{b}^2\right)^2 - 2a^2\mathrm{b}^2 = \left(x^2 - 2\mathrm{y}\right)^2 - 2\mathrm{y}^2 = x^4 - 4x^2\mathrm{y} + 2\mathrm{y}^2$

$$\overset{\text{via }(\bullet)}{=} x^4 - 4x^2.\frac{x^3+x^2-4}{3x+2} + 2\left(\frac{x^3+x^2-4}{3x+2}\right)^2 \geq 2$$

$$\Leftrightarrow \frac{x^6+4x^5+2x^4-32x^3+2x^2+24x-24}{(3x+2)^2} \leq 0$$

$\Leftrightarrow x^6+4x^5+2x^4-32x^3+2x^2+24x-24 \leq 0$

$\Leftrightarrow (x-2)\left(x^5+6x^4+14x^3-4x^2-6x+12\right) \leq 0$ and so, it suffices to prove :

$$x^5+6x^4+14x^3-4x^2-6x+12 \overset{(*)}{>} 0\ \left(\because x-2 \leq 0 \text{ via } (1)\right) \text{ and}$$

$\because x^4 - 4x^2 + 4 = \left(x^2-2\right)^2 \geq 0 \therefore$ in order to prove $(*)$, it suffices to prove :

$$x^5+5x^4+14x^3-6x+8 > 0 \Leftrightarrow 81x^5+405x^4+1134x^3-486x+648 > 0$$

$$\Leftrightarrow (3x-1)^2\left(9x^3+51x^2+159x+\frac{301}{3}\right) + \frac{1643}{3} - 43x \overset{(**)}{>} 0$$

$$\text{Now, } \frac{1643}{3} - 43x \overset{\text{via }(1)}{\geq} \frac{1643}{3} - 86 > 0 \Rightarrow \frac{1643}{3} - 43x > 0 \to \text{(i) and also,}$$

$$9x^3 + 6x^2 = 3x^2(3x+2) \geq 0 \to \text{(ii)} \left(\because 3x+2 \overset{\text{via }(1)}{>} 0\right) \therefore \text{(i), (ii)} \Rightarrow$$

in order to prove $(**)$, it suffices to prove :

$$45x^2 + 159x + \frac{301}{3} > 0 \Leftrightarrow 135x^2 + 477x + 301 > 0$$

$$\Leftrightarrow (3x+2)(15(3x+2)+99)+43 > 0 \to \text{true} \because 3x+2 \overset{\text{via }(1)}{>} 0 \Rightarrow (**) \Rightarrow (*)$$

is true $\therefore a^4 + \mathrm{b}^4 \geq 2\ \forall\, a, \mathrm{b} \in \mathbb{R} \,\middle|\, a^3+\mathrm{b}^3+a^2+\mathrm{b}^2 = 4, '' ='' \text{ iff } a = \mathrm{b} = 1$ (QED)

1285. If $a, b, c > 0$ and $a+b+c=3$, then prove that :

$$\frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2b^2+1}}+\frac{1}{\sqrt{2c^2+1}}\ge\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c} \text{ and then : } \sum_{cyc}\frac{1}{x}\overset{a+b+c=3}{=}3\to(1)$$

$$\left(\frac{1}{x}=a<3\Rightarrow x>\frac{1}{3}\text{ and analogs}\right)$$

$$\text{We shall now prove that : }\frac{x^2}{\sqrt{x^2+2}}\ge\frac{5x-2}{3\sqrt{3}}\ \forall\, x\in\left(\frac{1}{3},\infty\right)$$

$$\text{If } \frac{1}{3}<x\le\frac{2}{5}, \text{ then : }\frac{5x-2}{3\sqrt{3}}\le 0<\frac{x^2}{\sqrt{x^2+2}}\text{ and so, we now focus on } x>\frac{2}{5}$$

$$\text{and then : }\frac{x^2}{\sqrt{x^2+2}}\ge\frac{5x-2}{3\sqrt{3}}\Leftrightarrow\frac{x^4}{x^2+2}\ge\frac{(5x-2)^2}{27}\Leftrightarrow 27x^4\ge(x^2+2)(5x-2)^2$$

$$\Leftrightarrow x^4+10x^3-27x^2+20x-4\ge 0\Leftrightarrow(x-1)^2\left(x^2+4(3x-1)\right)\ge 0\to\text{true}$$

$$\therefore\frac{x^2}{\sqrt{x^2+2}}\ge\frac{5x-2}{3\sqrt{3}}\ \forall\, x\in\left(\frac{1}{3},\infty\right)\text{ and analogs}\to(2)$$

$$\text{Now, }\frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2b^2+1}}+\frac{1}{\sqrt{2c^2+1}}=\sum_{cyc}\frac{1}{\sqrt{\frac{2}{x^2}+1}}=\sum_{cyc}\left(\frac{1}{x}\cdot\frac{x^2}{\sqrt{x^2+2}}\right)$$

$$\overset{\text{via }(2)}{\ge}\sum_{cyc}\left(\frac{1}{x}\cdot\frac{5x-2}{3\sqrt{3}}\right)=\frac{5}{\sqrt{3}}-\frac{2}{3\sqrt{3}}\cdot\sum_{cyc}\frac{1}{x}\overset{\text{via }(1)}{=}\frac{5}{\sqrt{3}}-\frac{6}{3\sqrt{3}}=\sqrt{3}$$

$$\therefore\frac{1}{\sqrt{2a^2+1}}+\frac{1}{\sqrt{2b^2+1}}+\frac{1}{\sqrt{2c^2+1}}\ge\sqrt{3}$$

$$\forall\, a,b,c>0\,\big|\,a+b+c=3,\text{''}=\text{'' iff } a=b=c=1\text{ (QED)}$$

Solution 2 by Tapas Das-India

Lemma:

If $x\in(0,3)$ then:

$$\frac{1}{\sqrt{2x^2+1}}\ge\frac{5-2x}{3\sqrt{3}}$$

Proof:

$$\frac{1}{\sqrt{2x^2+1}} \geq \frac{5-2x}{3\sqrt{3}} \Leftrightarrow (5-2x)^2(2x^2+1) \leq 27$$

$$\Leftrightarrow\ 8x^4 - 40x^3 + 54x^2 - 20x - 2 \leq 0 \Leftrightarrow (x-1)^2(4x^2-12x-1) \leq 0$$

$$\Leftrightarrow (x-1)^2[(2x-3)^2-10] \leq 0\ \textit{true}.\ (as\ 2x-3<3)$$

$$\frac{1}{\sqrt{2a^2+1}} + \frac{1}{\sqrt{2b^2+1}} + \frac{1}{\sqrt{2c^2+1}} \overset{\text{Lemma}}{\geq} \sum_{cyc} \frac{5-2a}{3\sqrt{3}} = \frac{15-2\cdot 3}{3\sqrt{3}} = \sqrt{3}$$

Equality holds for $a=b=c=1$.

1286. **If $a, b, c > 0$, *then prove that* :**

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} + 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} + 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = \sum_{cyc}\left(\frac{a^3}{2bc} + \frac{a^3}{2bc} + \frac{4}{a^2}\right) \overset{A-G}{\geq} 3\sum_{cyc}\sqrt[3]{\frac{a^4}{b^2c^2}}$$

$$= \frac{3}{\sqrt[3]{a^2b^2c^2}}\sum_{cyc} a^2 \overset{A-G}{\geq} \frac{9\sqrt[3]{a^2b^2c^2}}{\sqrt[3]{a^2b^2c^2}}. = 9, \therefore \frac{a^3}{bc} + \frac{b^3}{ca} + \frac{c^3}{ab} + 4\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geq 9$$

$$\forall\, a, b, c > 0, '' ='' \text{ iff } a=b=c \text{ and for } a^5 = 8a^2 \text{ and analogs}$$

$$\Rightarrow \text{iff } a=b=c=2\ (\text{QED})$$

1287. **If $a, b, c \in \mathbb{R}$ and $(a+1)(b+1)(c+1) = 8$, then prove that :**

$$a^2 + b^2 + c^2 \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

We assert that $\forall\, x, y, z \geq 0$, we have : $\sqrt[3]{xyz} \leq \frac{x+y+z}{3} \to (1)$

When $x=y=z=0$, then trivially (1) is true. When exactly two among x, y, z

$= 0$ and WLOG we may assume $y = z = 0$ $(x > 0)$, then :

$$\frac{x+y+z}{3} = \frac{x}{3} > 0 = \sqrt[3]{xyz} \Rightarrow (1) \text{ is true}$$

When exactly one among $x, y, z = 0$ and WLOG we may assume $x = 0$ $(y, z > 0)$,

then : $\frac{x+y+z}{3} = \frac{y+z}{3} > 0 = \sqrt[3]{xyz} \Rightarrow (1)$ is true. When $x, y, z > 0$, (1) is true

by AM − GM and so, $\forall\, x, y, z \geq 0$, we have : $\sqrt[3]{xyz} \leq \frac{x+y+z}{3}$

Now, $(a+1)(b+1)(b+1) = 8 \Rightarrow (a+1)^2(b+1)^2(c+1)^2 = 64 \Rightarrow 4$

$$= \sqrt[3]{(a+1)^2(b+1)^2(c+1)^2} \overset{\text{via }(1)}{\leq} \frac{(a+1)^2+(b+1)^2+(c+1)^2}{3}$$

$$\Rightarrow \sum_{\text{cyc}} a^2 + 2\sum_{\text{cyc}} a \geq 9 \to (2)$$

Case 1 $\sum_{\text{cyc}} a \leq 3$ and so, via (2), $\sum_{\text{cyc}} a^2 \geq 9 - 2\sum_{\text{cyc}} a \geq 9 - 6 \Rightarrow a^2+b^2+c^2 \geq 3$

Case 2 $\sum_{\text{cyc}} a \geq 3$ and then : $\sum_{\text{cyc}} a^2 \geq \frac{1}{3}\left(\sum_{\text{cyc}} a\right)^2 \left(\Leftrightarrow \sum_{\text{cyc}} (a-b)^2 \geq 0 \to \text{true}\right)$

$\geq \frac{9}{3} \Rightarrow a^2+b^2+c^2 \geq 3$ ∴ combining both cases, $a^2+b^2+c^2 \geq 3$

$\forall\, a, b, c \in \mathbb{R} \mid (a+1)(b+1)(c+1) = 8$ (QED)

1288. If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{a}{b^{2024}} + \frac{b}{c^{2024}} + \frac{c}{a^{2024}} \geq a+b+c$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b^{2024}} + \frac{b}{c^{2024}} + \frac{c}{a^{2024}} = \frac{\left(\frac{1}{b}\right)^{2024}}{\frac{1}{a}} + \frac{\left(\frac{1}{c}\right)^{2024}}{\frac{1}{b}} + \frac{\left(\frac{1}{a}\right)^{2024}}{\frac{1}{c}} \overset{\text{Holder}}{\geq} \frac{\left(\sum_{\text{cyc}} \frac{1}{a}\right)^{2024}}{3^{2022}\left(\sum_{\text{cyc}} \frac{1}{a}\right)}$$

$$= \frac{\left(\sum_{\text{cyc}} \frac{1}{a}\right)^2 \left(\sum_{\text{cyc}} \frac{1}{a}\right)^{2021}}{3^{2022}} \overset{A-G}{\geq} \frac{\frac{\left(\sum_{\text{cyc}} ab\right)^2}{a^2b^2c^2} \cdot \left(3\sqrt[3]{\frac{1}{abc}}\right)^{2021}}{3^{2022}} \geq \frac{\frac{3abc\sum_{\text{cyc}} a}{a^2b^2c^2}}{3} \quad (\because abc = 1)$$

$$= \frac{\sum_{\text{cyc}} a}{abc} \overset{abc=1}{=} \sum_{\text{cyc}} a \therefore \frac{a}{b^{2024}} + \frac{b}{c^{2024}} + \frac{c}{a^{2024}} \geq a+b+c$$

$\forall\, a, b, c > 0 \mid abc = 1$, "=" iff $a = b = c = 1$ (QED)

1289. If $a, b, c > 0$ and $a^2 + b^2 + c^2 \leq 192$, then prove that :

$$\sqrt{a^3 + 64} + \sqrt{b^3 + 64} + \sqrt{b^3 + 64} \leq 72$$

Proposed by Nguyen Hung Cuong

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{a^3 + 64} + \sqrt{b^3 + 64} + \sqrt{b^3 + 64} = \sum_{cyc} \sqrt{(a+4)(a^2 - 4a + 16)}$$

$$\overset{CBS}{\leq} \sqrt{\sum_{cyc}(a+4)} \cdot \sqrt{\sum_{cyc}(a^2 - 4a + 16)} = \sqrt{t+12} \cdot \sqrt{\sum_{cyc} a^2 - 4t + 48} \left(t = \sum_{cyc} a\right)$$

$$\overset{a^2+b^2+c^2 \leq 192}{\leq} \sqrt{(t+12)(240-4t)} \left(\text{note} : 240 - 4t \overset{CBS}{\geq} 240 - 4 \cdot \sqrt{3 \sum_{cyc} a^2} \overset{a^2+b^2+c^2 \leq 192}{\geq} 240 - 96 > 0\right)$$

$$\overset{?}{\leq} 72 \Leftrightarrow t^2 - 48t + 576 \overset{?}{\geq} 0 \Leftrightarrow (t-24)^2 \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore \sqrt{a^3 + 64} + \sqrt{b^3 + 64} + \sqrt{b^3 + 64} \leq 72$$

$$\forall\, a, b, c > 0 \text{ and } a^2 + b^2 + c^2 \leq 192 \text{ (QED)}$$

Solution 2 by Tapas Das-India

Let $a^2 = p, b^2 = q, c^2 = r$, *and* $p + q + r \leq 192$ *and* $\sum \sqrt{a^3 + 64} = \sum \sqrt{p^{\frac{3}{2}} + 64}$,

now we will show as a lemma that $\frac{32 + p}{4} \geq \sqrt{p^{\frac{3}{2}} + 64}$ *or*,

$$(32 + p)^2 \geq 16\left(p^{\frac{3}{2}} + 64\right) \text{ or } u^2 - 16u + 64 \overset{p=u^2}{\geq} 0 \text{ or } (u-8)^2 \geq 0 (True).$$

back to the main problem:

$$LHS \overset{lemma}{\leq} \sum \frac{32 + p}{4} = \frac{96 + p + q + r}{4} \overset{\sum p \leq 192}{\leq} = \frac{96 + 192}{4} = 72$$

1290. If $a, b, c > 0, (a+1)(b+1)(c+1) = 8$ then:

$$a^2 + b^2 + c^2 \geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\textit{Let } 1 + a = x, \quad 1 + b = y, \quad 1 + c = z$$

$$\textit{Then } xyz = 8.$$

$$x+y+z\geq 3\sqrt[3]{xyz}=3\cdot 2=6$$

$$(x-1)^2+(y-1)^2+(z-1)^2\geq 3\ (to\ prove)$$

$$\frac{(x-1)^2}{1}+\frac{(y-1)^2}{1}+\frac{(z-1)^2}{1}\overset{Bergstrom}{\geq}\frac{(x+y+z-3)^2}{1+1+1}\geq\frac{(6-3)^2}{3}=3$$

Equality holds for $a=b=c=1$.

1291. If $a,b>0$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2}\geq\frac{24(a^2+b^2)}{(a+b)^4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+b^2+2ab)^2\overset{A-G}{\geq}8ab(a^2+b^2)\Rightarrow\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2}-\frac{24(a^2+b^2)}{(a+b)^4}$$

$$\geq\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2}-\frac{24(a^2+b^2)}{8ab(a^2+b^2)}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{2}{ab}-\left(\frac{1}{ab}-\frac{2}{a^2+b^2}\right)$$

$$=\frac{a^2+b^2-2ab}{a^2b^2}-\frac{a^2+b^2-2ab}{ab(a^2+b^2)}=\frac{(a-b)^2(a^2+b^2-ab)}{a^2b^2(a^2+b^2)}\overset{A-G}{\geq}$$

$$\frac{(a-b)^2(2ab-ab)}{a^2b^2(a^2+b^2)}=\frac{(a-b)^2}{ab(a^2+b^2)}\geq 0\therefore\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2}\geq\frac{24(a^2+b^2)}{(a+b)^4}$$

$$\forall\, a,b>0,''=''\ \text{iff}\ a=b\ (\text{QED})$$

1292. If $a,b,c>0$, $ab+bc+ca=abc$ then:

$$\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(c^3+a^3)}\geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\frac{a^4+b^4}{ab(a^3+b^3)}+\frac{b^4+c^4}{bc(b^3+c^3)}+\frac{c^4+a^4}{ac(c^3+a^3)}\overset{CEBYSHEV}{\geq}$$

$$\geq \sum \frac{1}{2}\frac{(a+b)(a^3+b^3)}{ab(a^3+b^3)} \geq \frac{1}{2}\sum \frac{a+b}{ab} = \sum \frac{1}{a} = \frac{\sum ab}{abc} = 1$$

Equality holds for: $a = b = c$.

1293. If $a, b > 0$ then:

$$\sqrt{a^2+\frac{1}{b^2}} + \sqrt{b^2+\frac{1}{a^2}} \geq 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sqrt{a^2+\frac{1}{b^2}} + \sqrt{b^2+\frac{1}{a^2}} \geq$$

$$\overset{AM-GM}{\geq} \sqrt{\frac{2a}{b}} + \sqrt{\frac{2b}{a}} \overset{AM-GM}{\geq} 2\sqrt{\sqrt{\left(\frac{2a}{b}\right).\left(\frac{2b}{a}\right)}} = 2\sqrt{2}$$

Equality holds for $a = b$.

1294. If $a, b, c > 0, a^2+b^2+c^2 = abc$ then:

$$\frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab} \leq \frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Tapas Das-India

$$\frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab} \overset{AM-HM}{\leq} \frac{1}{4}\sum\left(\frac{a}{a^2}+\frac{a}{bc}\right) = \frac{1}{4}\sum\frac{1}{a} + \frac{1}{4}\frac{a^2+b^2+c^2}{abc}$$

$$= \frac{1}{4}\frac{ab+bc+ca}{abc} + \frac{1}{4} \leq \frac{1}{4}\frac{\sum a^2}{abc} + \frac{1}{4} = \frac{1}{2}\ \left(since \sum a^2 = abc\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{LHS} = \sum_{\text{cyc}} \frac{a^2+\text{bc}-\text{bc}}{a(a^2+\text{bc})} = \sum_{\text{cyc}} \frac{1}{a} - \sum_{\text{cyc}} \frac{\text{b}^2\text{c}^2}{a^3\text{bc}+\text{ab}^2\text{c}^2} \overset{\text{Bergstrom}}{\leq}$$

$$\frac{\sum_{cyc} ab}{abc}-\frac{\left(\sum_{cyc} ab\right)^2}{abc\sum_{cyc} a^2+abc\sum_{cyc} ab}\le\frac{\sum_{cyc} ab}{abc}-\frac{\left(\sum_{cyc} ab\right)^2}{abc\sum_{cyc} a^2+abc\sum_{cyc} a^2}$$

$$\overset{a^2+b^2+c^2=abc}{=}\frac{\sum_{cyc} ab}{abc}-\frac{\left(\sum_{cyc} ab\right)^2}{2a^2b^2c^2}=\frac{2abc\sum_{cyc} ab-\left(\sum_{cyc} ab\right)^2}{2a^2b^2c^2}\Rightarrow \text{LHS}-\text{RHS}\le$$

$$\frac{2abc\sum_{cyc} ab-\left(\sum_{cyc} ab\right)^2}{2a^2b^2c^2}-\frac{1}{2}=\frac{2abc\sum_{cyc} ab-\left(\sum_{cyc} ab\right)^2-a^2b^2c^2}{2a^2b^2c^2}$$

$$=-\frac{1}{2a^2b^2c^2}\cdot\left(abc-\sum_{cyc} ab\right)^2=\frac{-1}{2a^2b^2c^2}\cdot\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)^2\le 0$$

$$\therefore \frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\le\frac{1}{2}$$

$$\forall\, a,b,c>0 \,\middle|\, a^2+b^2+c^2=abc,''=''\ \text{iff } a=b=c=3\ \text{(QED)}$$

1295. If $a,b,c>0$ and $ab+bc+ca=3$, then prove that :

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\le\frac{1}{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3=ab+bc+ca\overset{A-G}{\ge}3\sqrt[3]{a^2b^2c^2}\Rightarrow abc\le 1\to\text{(i)}$$

$$\text{Now, LHS}=\sum_{cyc}\frac{1}{1+a(3-bc)}\ (\because a(b+c)=3-bc\ \text{and analogs})$$

$$=\sum_{cyc}\frac{1}{1-abc+3a}\overset{\text{via (i)}}{\le}\sum_{cyc}\frac{1}{3a}=\frac{1}{3abc}\cdot\sum_{cyc} ab=\frac{3}{3abc}$$

$$\therefore \frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\le\frac{1}{abc}$$

$$\forall\, a,b,c>0 \,\middle|\, ab+bc+ca=3,''=''\ \text{iff } a=b=c=1\ \text{(QED)}$$

1296.

If $a, b, c \geq 0$ and $a+b+c=3$, then prove that :

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\right)^2\geq$$

$$\sum_{cyc}a+\sum_{cyc}(b-c)^2+2\sum_{cyc}\sqrt{ab}=\sum_{cyc}a+2\sum_{cyc}a^2-2\sum_{cyc}ab+2\sum_{cyc}\sqrt{ab}\overset{?}{\geq}9$$

$$\Leftrightarrow\sum_{cyc}x^2+2\sum_{cyc}x^4-2\sum_{cyc}x^2y^2+2\sum_{cyc}xy\overset{?}{\geq}9\ (\sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z)$$

$$\Leftrightarrow\frac{1}{3}\left(\sum_{cyc}x^2\right)^2+2\sum_{cyc}x^4-2\sum_{cyc}x^2y^2+\frac{2}{3}\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)\overset{?}{\geq}\left(\sum_{cyc}x^2\right)^2$$

$$\left(\because\sum_{cyc}x^2=3\right)\Leftrightarrow\sum_{cyc}x^4+2\sum_{cyc}x^2y^2+6\sum_{cyc}x^4-6\sum_{cyc}x^2y^2$$

$$+2\left(\sum_{cyc}x^3y+\sum_{cyc}xy^3+xyz\sum_{cyc}x\right)\overset{?}{\geq}3\sum_{cyc}x^4+6\sum_{cyc}x^2y^2$$

$$\Leftrightarrow 2\sum_{cyc}x^4-5\sum_{cyc}x^2y^2+\sum_{cyc}x^3y+\sum_{cyc}xy^3+xyz\sum_{cyc}x\underset{(*)}{\overset{?}{\geq}}0$$

Now, via Schur, $\sum_{cyc}x^4+xyz\sum_{cyc}x\geq\sum_{cyc}x^3y+\sum_{cyc}xy^3\Rightarrow$ LHS of $(*)\geq$

$$\sum_{cyc}x^4+2\sum_{cyc}x^3y+2\sum_{cyc}xy^3-5\sum_{cyc}x^2y^2\geq\sum_{cyc}x^4+4\sum_{cyc}x^2y^2-5\sum_{cyc}x^2y^2$$

$(\because x^3y+xy^3-2x^2y^2=xy(x-y)^2\geq 0\Rightarrow x^3y+xy^3\geq 2x^2y^2$ and analogs$)$

$$=\sum_{cyc}x^4-\sum_{cyc}x^2y^2=\frac{1}{2}\sum_{cyc}(x^2-y^2)^2\geq 0\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\right)^2\geq 9$$

$$\Rightarrow\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\geq 3$$

$\forall\, a,b,c>0\,|\,a+b+c=3,$ "=" iff $a=b=c=1$ (QED)

1297. If $a, b, c > 0$ and $abc \geq 1$, then prove that :

$$a + b + c \geq \frac{1+a}{1+b} + \frac{1+b}{1+c} + \frac{1+c}{1+a}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$a + b + c \geq \frac{1+a}{1+b} + \frac{1+b}{1+c} + \frac{1+c}{1+a}$$

$$\Leftrightarrow a\left(1 - \frac{1}{1+b}\right) - \frac{1}{1+b} + b\left(1 - \frac{1}{1+c}\right) - \frac{1}{1+c} + c\left(1 - \frac{1}{1+a}\right) - \frac{1}{1+a} \geq 0$$

$$\Leftrightarrow \frac{ab-1}{1+b} + \frac{bc-1}{1+c} + \frac{ca-1}{1+a} \geq 0 \Leftrightarrow$$

$$(ab-1)(1+c)(1+a) + (bc-1)(1+a)(1+b) + (ca-1)(1+b)(1+c) \geq 0$$

$$\Leftrightarrow abc\sum_{cyc} a + 3abc - 3 + \sum_{cyc} a^2b - 2\sum_{cyc} a \overset{(*)}{\geq} 0$$

$$\because abc \geq 1 \therefore abc\sum_{cyc} a - \sum_{cyc} a + 3abc - 3 \geq 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : } \sum_{cyc} a^2b \overset{(**)}{\geq} \sum_{cyc} a$$

$\boxed{\text{Case 1}}$ $\sum_{cyc} a \geq \sum_{cyc} ab$ and then : $\sum_{cyc} a^2b = \sum_{cyc} \frac{a^2}{\frac{1}{b}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} \frac{1}{a}}$

$$= abc.\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} ab} \overset{\substack{\sum_{cyc} ab \leq \sum_{cyc} a \\ \text{and} \\ \because abc \geq 1}}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a} = \sum_{cyc} a \Rightarrow \sum_{cyc} a^2b \geq \sum_{cyc} a$$

$\boxed{\text{Case 2}}$ $\sum_{cyc} ab \geq \sum_{cyc} a$ and then : $\sum_{cyc} a^2b = \sum_{cyc} \frac{a^2b^2}{b} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} ab\right)^2}{\sum_{cyc} a}$

$$\overset{\sum_{cyc} ab \geq \sum_{cyc} a}{\geq} \frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a} = \sum_{cyc} a \Rightarrow \sum_{cyc} a^2b \geq \sum_{cyc} a \therefore \text{combining both cases,}$$

$$(**) \Rightarrow (*) \text{ is true } \forall\, a, b, c > 0 \,\big|\, abc \geq 1 \Rightarrow a + b + c \geq \frac{1+a}{1+b} + \frac{1+b}{1+c} + \frac{1+c}{1+a}$$

$$\forall\, a, b, c > 0 \,\big|\, abc \geq 1, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1298. If $a, b, c > 0$ and $a^2 + b^2 + c^2 \geq 3$, then prove that :

$$\frac{a^2}{b+2c} + \frac{b^2}{c+2a} + \frac{c^2}{a+2b} \geq 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{b+2c} + \frac{b^2}{c+2a} + \frac{c^2}{a+2b} = \frac{a^4}{a^2b+2ca^2} + \frac{b^4}{b^2c+2ab^2} + \frac{c^4}{c^2a+2bc^2}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} a^2b + \sum_{cyc} ab^2 + \sum_{cyc}\left(a\left(\sum_{cyc} a^2 - c^2 - a^2\right)\right)}$$

$$= \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} a^2b + \sum_{cyc} ab^2 + \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) - \sum_{cyc} a^2b - \sum_{cyc} a^3} \overset{\sum_{cyc} ab^2 \leq \sum_{cyc} a^3}{\geq}$$

$$\frac{\left(\sum_{cyc} a^2\right)^2}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right)} = \frac{\sum_{cyc} a^2}{\sum_{cyc} a} = \frac{\sqrt{\frac{\sum_{cyc} a^2}{3}}\cdot\sqrt{3\sum_{cyc} a^2}}{\sum_{cyc} a} \overset{\substack{a^2+b^2+c^2 \geq 3 \\ \text{and} \\ 3\sum_{cyc} a^2 \geq \left(\sum_{cyc} a\right)^2}}{\geq} = 1$$

$$\therefore \frac{a^2}{b+2c} + \frac{b^2}{c+2a} + \frac{c^2}{a+2b} \geq 1\ \forall\ a, b, c > 0 \mid a^2 + b^2 + c^2 \geq 3,$$
$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$

1299. If $a, b, c \geq 0$ and $a + b + c = 1$, then prove that :

$$ab + bc + ca \geq 8(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)$$

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Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 among a, b, c equal zero and WLOG we may assume $b = c = 0$ $(a = 1)$ and then : LHS = RHS = 0

$\boxed{\text{Case 2}}$ Exactly 1 among a, b, c equals zero and WLOG we may assume $a = 0$ $(b + c = 1$ with $b, c > 0)$ and then : $\text{LHS} - \text{RHS} \overset{b+c=1}{=}$

$$bc(b+c)^4 - 8b^2c^2(b^2+c^2) = bc\left((b^2+c^2+2bc)^2 - 8bc(b^2+c^2)\right)$$

$$= bc\left((b^2+c^2)^2 + 4b^2c^2 - 4bc(b^2+c^2)\right) = bc(b^2+c^2-2bc)^2 = bc(b-c)^4 \geq 0$$

$$\Rightarrow \text{LHS} \geq \text{RHS}$$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow$ $x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z,$

$y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $=$ s, R, r (say); so $2\sum_{cyc} a = \sum_{cyc} x = 2s$

$$\Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s-x, b = s-y, c = s-z \therefore abc = r^2 s \to (2)$$

and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3)$,

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2) \to (5)$$

$$\therefore ab + bc + ca \ge 8(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2) \overset{a+b+c=1}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a\right)^4\left(\sum_{cyc} ab\right) \ge 8\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)$$

$$\overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow} (4Rr + r^2)s^4 \ge 8r^2(s^2 - 8Rr - 2r^2)((4R+r)^2 - 2s^2)$$

$$\Leftrightarrow (4R+17r)s^4 - rs^2(128R^2 + 192Rr + 40r^2) + 16r^2(4R+r)^3 \overset{(*)}{\ge} 0 \text{ and}$$

$$\because (4R+17r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{in order to prove } (*),$$

it suffices to prove : LHS of $(*) \ge (4R+17r)(s^2 - 16Rr + 5r^2)^2$

$$\Leftrightarrow (312R - 210r)s^2 \overset{(**)}{\ge} r(2944R^2 - 2812Rr + 409r^2)$$

$$\text{Now, } (312R - 210r)s^2 \overset{\text{Gerretsen}}{\ge} (312R - 210r)(16Rr - 5r^2) \overset{?}{\ge}$$

$$r(2944R^2 - 2812Rr + 409r^2) \Leftrightarrow 2048R^2 - 2108Rr + 641r^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow 994R^2 + 1054R(R-2r) + 641r^2 \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$\therefore ab + bc + ca > 8(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)$ and combining *all* cases,

$ab + bc + ca \ge 8(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)\ \forall\, a, b, c \ge 0 \mid a+b+c = 1$,

$'' ='' $ iff $(a = 1, b = c = 0)$ and permutations and $\left(a = 0, b = c = \frac{1}{2}\right)$

and permutations (QED)

1300. If $a, b > 0$ and $a^2 + b^2 \geq 2$, then prove that :

$$\frac{a^2}{b} + \frac{b^2}{a} + 7(a + b) \geq 16$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = a + b \text{ and } y = ab \therefore a^2 + b^2 \geq 2 \Leftrightarrow x^2 - 2y \geq 2$$
$$\Leftrightarrow 2y \leq x^2 - 2 \to (\text{i})$$
$$\text{Now, } \frac{a^2}{b} + \frac{b^2}{a} + 7(a + b) = \frac{(a + b)(a^2 - ab + b^2)}{ab} + 7(a + b) = \frac{x(x^2 - 3y)}{y} + 7x$$
$$= \frac{x^3 + 4xy}{y} \overset{?}{\geq} 16 \Leftrightarrow x^3 \underset{(*)}{\overset{?}{\geq}} 4y(4 - x)$$

$\boxed{\text{Case 1}}$ $4 - x \leq 0$ and $\because x, y > 0 \therefore LHS\ of\ (*) > 0$ and RHS of $(*) \leq 0 \Rightarrow$ LHS of $(*) >$ RHS of $(*)$

$\boxed{\text{Case 2}}$ $4 - x > 0$ and then : $4y(4 - x) \overset{\text{via (i)}}{\leq} (8 - 2x)(x^2 - 2) \overset{?}{\leq} x^3 \Leftrightarrow$
$3x^3 - 8x^2 - 4x + 16 \overset{?}{\geq} 0 \Leftrightarrow (3x + 4)(x - 2)^2 \overset{?}{\geq} 0 \to$ true $\because x = a + b > 0$
$\Rightarrow (*)$ is true $\therefore$ combining both cases, $(*)$ is true $\forall\, x, y > 0$ constrained by (i)

$$\therefore \frac{a^2}{b} + \frac{b^2}{a} + 7(a + b) \geq 16\ \forall\, a, b > 0 \,\middle|\, a^2 + b^2 \geq 2 \text{ (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru