ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c\in\mathbb{R}$ and $abc(a^3+b^3+c^3)\geq 3$, then prove that :

$$a^2+b^2+c^2\geq 3$$

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$$abc(a^3+b^3+c^3)\geq 3\Rightarrow a^2b^2c^2\left(\sum_{cyc}a^3\right)^2\geq 9$$

$$\Rightarrow\sum_{cyc}a^6+2\sum_{cyc}a^3b^3\geq\frac{9}{a^2b^2c^2}\to(1)$$

$$\boxed{\text{Case 1}}\ a^2b^2c^2\geq 1 \text{ and then : }\left(\sum_{cyc}a^2\right)^3$$

$$=\sum_{cyc}a^6+3\left(2a^2b^2c^2+\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)$$

$$=\sum_{cyc}a^6+\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+6a^2b^2c^2+2\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)$$

$$\overset{A-G}{\geq}\sum_{cyc}a^6+2\sum_{cyc}a^3b^3+6a^2b^2c^2+2(3a^2b^2c^2+3a^2b^2c^2)\overset{\text{via }(1)}{\geq}$$

$$\frac{9}{a^2b^2c^2}+18a^2b^2c^2\overset{?}{\geq}27\Leftrightarrow 2x^4-3x^2+1\overset{?}{\geq}0\ (x=abc)$$

$$\Leftrightarrow(x^2-1)(2x^2-1)\overset{?}{\geq}0\to\text{true}\because x^2-1\geq 0 \text{ and } 2x^2\geq 2>1\Rightarrow 2x^2-1>0$$

$$\therefore\left(\sum_{cyc}a^2\right)^3\geq 27\Rightarrow a^2+b^2+c^2\geq 3$$

$$\boxed{\text{Case 2}}\ a^2b^2c^2\leq 1 \text{ and then : }\left(\sum_{cyc}a^2\right)^3$$

$$=\sum_{cyc}a^6+3\left(2a^2b^2c^2+\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)$$

$$=\sum_{cyc}a^6+\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4+6a^2b^2c^2+2\left(\sum_{cyc}a^4b^2+\sum_{cyc}a^2b^4\right)$$

$$\overset{A-G}{\geq} \sum_{cyc} a^6 + 2\sum_{cyc} a^3b^3 + 6a^2b^2c^2 + 2\sum_{cyc}(a^4b^2 + a^4c^2) \overset{A-G}{\geq}$$

$$\sum_{cyc} a^6 + 2\sum_{cyc} a^3b^3 + 6a^2b^2c^2 + 4abc(a^3 + b^3 + c^3) \overset{\substack{\text{via } (1)\\ and\\ \because abc(a^3+b^3+c^3) \geq 3}}{\geq}$$

$$\frac{9}{a^2b^2c^2} + 6a^2b^2c^2 + 12 \overset{?}{\geq} 27 \Leftrightarrow 2x^4 - 5x^2 + 3 \overset{?}{\geq} 0 \Leftrightarrow (x^2 - 1)(2x^2 - 3) \overset{?}{\geq} 0$$

$$\to \text{true} \because x^2 - 1 \leq 0 \text{ and } 2x^2 \leq 2 < 3 \Rightarrow 2x^2 - 3 < 0 \Rightarrow (x^2 - 1)(2x^2 - 3) \leq 0$$

$$\therefore \left(\sum_{cyc} a^2\right)^3 \geq 27 \Rightarrow a^2 + b^2 + c^2 \geq 3 \therefore \text{combining both cases}, a^2 + b^2 + c^2 \geq 3$$

$$\forall\, a, b, c \in \mathbb{R} \,\middle|\, abc(a^3 + b^3 + c^3) \geq 3, '' ='' \text{ iff } a^2 = b^2 = c^2 \text{ and } a^2b^2c^2 = 1 \text{ and}$$

$$\because abc(a^3 + b^3 + c^3) \geq 3$$

$$\therefore \text{equality iff } (a = b = c = 1) \text{ or } (a = b = c = -1) \text{ (QED)}$$