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If $a, \mathrm{b}, \mathrm{c} \geq 0$ and $a + \mathrm{b} + \mathrm{c} = 1$, then prove that :

$$ab + \mathrm{bc} + \mathrm{c}a - 3a\mathrm{bc} \leq \frac{1}{4}$$

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$\boxed{\text{Case 1}}$ Exactly 2 among $a, \mathrm{b}, \mathrm{c}$ equal zero and WLOG we may assume

$$\mathrm{b} = \mathrm{c} = 0\ (a = 1) \text{ and then : LHS} = 0 < \frac{1}{4}$$

$\boxed{\text{Case 2}}$ Exactly 1 among $a, \mathrm{b}, \mathrm{c}$ equals zero and WLOG we may

$$\text{assume } a = 0\ (\mathrm{b} + \mathrm{c} = 1) \text{ and then : LHS} = \mathrm{bc} \overset{\mathrm{A-G}}{\leq} \frac{(\mathrm{b} + \mathrm{c})^2}{4} = \frac{1}{4}$$

$$\boxed{\text{Case 3}}\ a, \mathrm{b}, \mathrm{c} > 0 \text{ and then : LHS} \overset{a+\mathrm{b}+\mathrm{c}=1}{=} \left(\sum_{\mathrm{cyc}} a\mathrm{b}\right)\left(\sum_{\mathrm{cyc}} a\right) - 3a\mathrm{bc}$$

$$\overset{?}{\leq} \frac{1}{4} \overset{a+\mathrm{b}+\mathrm{c}=1}{=} \frac{1}{4}\left(\sum_{\mathrm{cyc}} a\right)^3$$

$$\Leftrightarrow \sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2 \overset{?}{\leq} \frac{1}{4}\left(\sum_{\mathrm{cyc}} a^3 + 3\left(2a\mathrm{bc} + \sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2\right)\right)$$

$$\Leftrightarrow \sum_{\mathrm{cyc}} a^3 + 6a\mathrm{bc} + 3\left(\sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2\right) \overset{?}{\geq} 4\left(\sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2\right)$$

$$\Leftrightarrow \sum_{\mathrm{cyc}} a^3 + 6a\mathrm{bc} \overset{?}{\geq} \sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2 \rightarrow \text{true}$$

$$\because \sum_{\mathrm{cyc}} a^3 + 6a\mathrm{bc} \overset{\mathrm{Schur}}{\geq} \sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2 + 3a\mathrm{bc} > \sum_{\mathrm{cyc}} a^2\mathrm{b} + \sum_{\mathrm{cyc}} a\mathrm{b}^2$$

$$\Rightarrow a\mathrm{b} + \mathrm{bc} + \mathrm{c}a - 3a\mathrm{bc} < \frac{1}{4} \text{ and combining all cases, } a\mathrm{b} + \mathrm{bc} + \mathrm{c}a - 3a\mathrm{bc} \leq \frac{1}{4}$$

$$\forall\, a, \mathrm{b}, \mathrm{c} \geq 0 \,\middle|\, a + \mathrm{b} + \mathrm{c} = 1, \text{''}=\text{''} \text{ iff } \left(a = 0, \mathrm{b} = \mathrm{c} = \frac{1}{2}\right)$$

$$\text{or } \left(\mathrm{b} = 0, \mathrm{c} = a = \frac{1}{2}\right) \text{ or } \left(\mathrm{c} = 0, a = \mathrm{b} = \frac{1}{2}\right) \text{ (QED)}$$