ROMANIAN MATHEMATICAL MAGAZINE

**If $a, b > 0$ and $a^3 + b^3 = ab$, then prove that :**

$$\frac{1}{a^6} + \frac{1}{b^6} \geq 128$$

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$$\frac{1}{a^6} + \frac{1}{b^6} = \frac{a^6 + b^6}{a^6 b^6} = \frac{(a^3 + b^3)^2 - 2a^3 b^3}{a^6 b^6} \overset{a^3 + b^3 = ab}{=} \frac{a^2 b^2 - 2a^3 b^3}{a^6 b^6}$$

$$= \frac{1 - 2ab}{a^4 b^4} \overset{?}{\geq} 128 \Leftrightarrow 128t^4 + 2t - 1 \overset{?}{\leq} 0 \ (t = ab)$$

$$\Leftrightarrow (4t - 1)(32t^3 + 8t^2 + 2t + 1) \overset{?}{\leq} 0 \rightarrow \text{true} \because ab = a^3 + b^3 \overset{A-G}{\geq} 2\sqrt{a^3 b^3}$$

$$\Rightarrow t^2 \geq 4t^3 \Rightarrow t \leq \frac{1}{4} \Rightarrow 4t - 1 \leq 0 \therefore \frac{1}{a^6} + \frac{1}{b^6} \geq 128$$

$$\forall\, a, b > 0 \,\middle|\, a^3 + b^3 = ab, '' = '' \text{ iff } a = b = \frac{1}{2} \text{ (QED)}$$