ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = 3$, then prove that :

$$\frac{4}{(a+b)^3}+\frac{4}{(b+c)^3}+\frac{4}{(c+a)^3} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{LHS}-\text{RHS} \overset{a+b+c=3}{=} \sum_{cyc}\frac{4}{(b+c)^3}-\sum_{cyc}\frac{3-(b+c)}{b+c}=\sum_{cyc}\left(\frac{4}{x^3}-\frac{3}{x}+1\right)$$

$$(x=b+c, y=c+a, z=a+b)=\sum_{cyc}\frac{x^3-3x^2+4}{x^3}=\sum_{cyc}\frac{(x-2)^2(x+1)}{x^3}\geq 0$$

$$\therefore \frac{4}{(a+b)^3}+\frac{4}{(b+c)^3}+\frac{4}{(c+a)^3} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

$$\forall\, a,b,c>0 \,\big|\, a+b+c=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Tapas Das-India

$$\frac{a}{b+c}=4.\frac{a}{b+c}.\frac{1}{2}.\frac{1}{2} \overset{AM-GM}{\leq} \frac{4}{27}\left(\frac{a}{b+c}+\frac{1}{2}+\frac{1}{2}\right)^3=$$

$$=\frac{4}{27}\left(\frac{a}{b+c}+1\right)^3=\frac{4}{27}\left(\frac{a+b+c}{b+c}\right)^3=4\left(\frac{1}{(b+c)^3}\right) \textit{ since } a+b+c=3.$$

$$\textit{similarly}, \frac{b}{c+a}\leq 4.\frac{1}{(c+a)^3} \textit{ and } \frac{c}{a+b}\leq 4.\frac{1}{(a+b)^3}$$

using this result we get

$$\frac{4}{(a+b)^3}+\frac{4}{(b+c)^3}+\frac{4}{(c+a)^3} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$