ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in [0,2]$, then prove that :

$$2(x+y+z)-(xy+yz+zx) \le 4$$

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$\boxed{\text{Case 1}}\ x = y = z = 0$ *and* then : LHS $= 0 < 4$

$\boxed{\text{Case 2}}$ Exactly 2 among x, y, z equal zero and WLOG we may assume $y = z = 0$ and $0 \le x \le 2$ and then : LHS $-$ RHS $= 2x - 4 = 2(x-2) \le 0$

$$\Rightarrow 2(x+y+z)-(xy+yz+zx) \le 4$$

$\boxed{\text{Case 3}}$ Exactly one among x, y, z equals zero and WLOG we may assume $x = 0$

and $0 \le y, z \le 2$ and then : $(y-2)(z-2) \ge 0 \Rightarrow yz - 2y - 2z + 4 \ge 0 \to (1)$

Also, LHS $-$ RHS $= 4 + yz - 2y - 2z \overset{\text{via }(1)}{\ge} 0 \Rightarrow 2(x+y+z)-(xy+yz+zx) \le 4$

$\boxed{\text{Case 4}}\ 0 < x, y, z \le 2$ and $\because 0 < x \le 2 \therefore \frac{1}{x} \ge \frac{1}{2} \Rightarrow \frac{1}{x} - \frac{1}{2} = a \text{ (say)} \ge 0 \Rightarrow x = \frac{2}{2a+1}$ and similarly, setting $\frac{1}{y} - \frac{1}{2} = b \ge 0$ and $\frac{1}{z} - \frac{1}{2} = c \ge 0$, we get :

$y = \frac{2}{2b+1}$ and $z = \frac{2}{2c+1}$ and then : $2(x+y+z)-(xy+yz+zx) \le 4 \Leftrightarrow$

$$4\sum_{\text{cyc}} \frac{1}{2a+1} - 4\sum_{\text{cyc}} \left(\frac{1}{2b+1} \cdot \frac{1}{2c+1}\right) \le 4$$

$$\Leftrightarrow \prod_{\text{cyc}} (2a+1) + \sum_{\text{cyc}} (2a+1) - \sum_{\text{cyc}} \big((2b+1)(2c+1)\big) \ge 0$$

$\Leftrightarrow 8abc + 1 \ge 0 \to$ true $\because a, b, c \ge 0 \therefore 2(x+y+z)-(xy+yz+zx) < 4$ and

$\therefore$ combining *all* cases, $2(x+y+z)-(xy+yz+zx) \le 4\ \forall\ x, y, z \in [0,2]$,

$'' ='' $ iff $(x = 2, y = z = 0)$ or $(y = 2, z = x = 0)$ or $(z = 2, x = y = 0)$ or $(x = y = 2, z = 0)$ or $(y = z = 2, x = 0)$ or $(z = x = 2, y = 0)$ (QED)