If $a, b, c > 0$, then prove that :

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\le\frac{3(a^2+b^2+c^2)}{a+b+c}$$

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Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a=\sum_{cyc} x=2s \Rightarrow \sum_{cyc} a=s \rightarrow (1) \Rightarrow a=s-x, b=s-y, c=s-z$ and

such substitutions $\Rightarrow \sum_{cyc} ab=\sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab=4Rr+r^2 \rightarrow (2)$,

$$\sum_{cyc} a^2=\left(\sum_{cyc} a\right)^2-2\sum_{cyc} ab \overset{\text{via }(1),(2)}{=} s^2-2(4Rr+r^2)$$

$$\Rightarrow \sum_{cyc} a^2=s^2-8Rr-2r^2 \rightarrow (3)$$

$$\text{Now, }\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\le\frac{3(a^2+b^2+c^2)}{a+b+c}$$

$$\Leftrightarrow \frac{3\sum_{cyc} a^2}{\sum_{cyc} a}\ge\sum_{cyc}\frac{(b+c)^2-2bc}{b+c} \Leftrightarrow \frac{3\sum_{cyc} a^2}{\sum_{cyc} a}+2\sum_{cyc}\frac{bc}{b+c}\ge 2\sum_{cyc} a \rightarrow (i)$$

$$\sum_{cyc}\frac{bc}{b+c}=\sum_{cyc}\frac{(s-y)(s-z)}{x}=\sum_{cyc}\frac{s^2-s(2s-x)+yz}{x}$$

$$=\frac{-s^2}{4Rrs}\sum_{cyc} xy+3s+\frac{1}{4Rrs}\sum_{cyc} x^2y^2$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2-s^2(s^2+4Rr+r^2)+12Rrs^2}{4Rrs}$$

$$=\frac{r(4R+r)^2+rs^2}{4Rs} \Rightarrow \frac{3\sum_{cyc} a^2}{\sum_{cyc} a}+2\sum_{cyc}\frac{bc}{b+c}-2\sum_{cyc} a$$

$$\overset{\text{via }(1)\text{ and }(3)}{=} \frac{3(s^2-8Rr-2r^2)}{s}+\frac{r(4R+r)^2+rs^2}{2Rs}-2s$$

$$=\frac{6R(s^2-8Rr-2r^2)+r(4R+r)^2+rs^2-4Rs^2}{2Rs}\overset{?}{\ge}0$$

$$\Leftrightarrow (2R+r)s^2 \underbrace{\overset{?}{\ge}}_{(*)} r(32R^2+4Rr-r^2)$$

$$\text{Now, }(2R+r)s^2 \overset{\text{Gerretsen}}{\ge} (2R+r)(16Rr-5r^2) \overset{?}{\ge} r(32R^2+4Rr-r^2)$$

$$\Leftrightarrow 2Rr-4r^2 \overset{?}{\ge} 0 \Leftrightarrow 2r(R-2r) \overset{?}{\ge} 0 \rightarrow \text{true via Euler} \Rightarrow (*) \Rightarrow (i)\text{ is true}$$

$$\therefore \frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\le\frac{3(a^2+b^2+c^2)}{a+b+c}$$

$\forall\, a, b, c > 0$, " = " iff $a=b=c$ (QED)