ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = 1$, then prove that :

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq 3(a^2+b^2+c^2)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via }(1),(3)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \rightarrow (4)$$

$$\text{Now, } \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} = \sum_{cyc}\left(\frac{a^2}{b} + b + 2a\right) - 3\sum_{cyc} a = \sum_{cyc}\frac{(a+b)^2}{b} - 3\sum_{cyc} a$$

$$= \sum_{cyc}\frac{\left((a+b)(a+c)\right)^2}{b(c+a)^2} - 3\sum_{cyc} a \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^2 + 3\sum_{cyc} ab\right)^2}{\sum_{cyc} bc^2 + \sum_{cyc} a^2 b + 6abc} - 3\sum_{cyc} a$$

$$= \frac{\left(\left(\sum_{cyc} a\right)^2 + \sum_{cyc} ab\right)^2}{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc + 6abc} - 3\sum_{cyc} a \overset{\text{via }(1),(2),(3)}{=} \frac{\left(s^2 + 4Rr + r^2\right)^2}{s(4Rr + r^2) + 3r^2 s} - 3s$$

$$= \frac{\left(s^2 + 4Rr + r^2\right)^2 - 3s^2(4Rr + r^2) - 9r^2 s^2}{s(4Rr + 4r^2)} \overset{?}{\geq} 3(a^2 + b^2 + c^2) \overset{a+b+c=1}{=} \frac{3\sum_{cyc} a^2}{\sum_{cyc} a}$$

$$\overset{\text{via }(1),(4)}{=} \frac{3(s^2 - 8Rr - 2r^2)}{s} \Leftrightarrow \left(s^2 + 4Rr + r^2\right)^2 - 3s^2(4Rr + r^2) - 9r^2 s^2$$

$$\overset{?}{\geq} 3(s^2 - 8Rr - 2r^2)(4Rr + 4r^2)$$

$$\Leftrightarrow s^4 - (16Rr + 22r^2)s^2 + r^2(112R^2 + 128Rr + 25r^2) \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$$

$$\text{Now, LHS of } (*) \overset{\text{Gerretsen}}{\geq} (16Rr - 5r^2)s^2 - (16Rr + 22r^2)s^2$$

$$+r^2(112R^2 + 128Rr + 25r^2) \overset{?}{\geq} 0 \Leftrightarrow 112R^2 + 128Rr + 25r^2 \boxed{\overset{?}{\underset{(**)}{\geq}}} 27s^2$$

$$\text{Again, } 27s^2 \overset{\text{Gerretsen}}{\leq} 108R^2 + 108Rr + 81r^2 \overset{?}{\leq} 112R^2 + 128Rr + 25r^2$$

$$\Leftrightarrow R^2+5Rr-14r^2\overset{?}{\geq}0\Leftrightarrow\left(R^2-4r^2\right)+5r(R-2r)\overset{?}{\geq}0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r$$

$$\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq3\left(a^2+b^2+c^2\right)$$

$$\forall a,b,c>0\,\middle|\,a+b+c=1,''=''\text{ iff }a=b=c=\frac{1}{3}\text{ (QED)}$$

Solution 2 by Tapas Das-India

$$(a+b+c)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\geq3\left(a^2+b^2+c^2\right)or$$

$$\sum\frac{a^3}{b}+\sum\frac{ab^2}{c}\geq2\sum a^2\ or\ \sum\frac{a^4}{ab}+\sum\frac{a^2b^2}{ac}\geq2\sum a^2\ or$$

$$\frac{\left(\sum a^2\right)^2}{\sum ab}+\frac{\left(\sum ab\right)^2}{\sum ab}-2\sum a^2\geq0(Bergstrom)\ or$$

$$\left(\sum a^2\right)^2-2\sum ab.\sum a^2+\left(\sum ab\right)^2\geq0\ or\left(\sum a^2-\sum ab\right)^2\geq0\ true$$

$$,Now\ \frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\frac{3\left(a^2+b^2+c^2\right)}{a+b+c}=3\left(a^2+b^2+c^2\right)(as\ a+b+c=1)$$