ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in [-2, 2]$, then prove that:

$$2(x^6 + y^6 + z^6) - x^4y^2 - y^4z^2 - z^4x^2 \le 192$$

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$-2 \le x \le 2 \Rightarrow (x+2)(x-2) \le 0 \Rightarrow x^2 - 4 \le 0 \Rightarrow 4 - x^2 \ge 0$ and similarly, $4 - y^2 \ge 0$ and $4 - z^2 \ge 0$ and setting $a = 4 - x^2, b = 4 - y^2$, $c = 4 - z^2$, we notice that : $\boxed{0 \le a, b, c \le 4}$ $(\because a = 4 - x^2 \le 4 \text{ and analogs})$ and $x^2 = 4 - a, y^2 = 4 - b, z^2 = 4 - c$ and via such substitutions,

$$2(x^6 + y^6 + z^6) - x^4y^2 - y^4z^2 - z^4x^2 \le 192$$

$$\Leftrightarrow 2\sum_{\text{cyc}}(4-a)^3 - \sum_{\text{cyc}}\left((4-a)^2(4-b)\right) - 192 \le 0$$

$$\Leftrightarrow \sum_{\text{cyc}} a^2b + 20\sum_{\text{cyc}} a^2 \overset{(*)}{\le} 2\sum_{\text{cyc}} a^3 + 8\sum_{\text{cyc}} ab + 48\sum_{\text{cyc}} a$$

Now, $(a-4)^2 \ge 0 \Rightarrow a^2 + 16 \ge 8a \Rightarrow 2a^3 + 32a \ge 16a^2$ $(\because a \ge 0)$ and analogs

$$\Rightarrow 2\sum_{\text{cyc}} a^3 + 32\sum_{\text{cyc}} a \ge 16\sum_{\text{cyc}} a^2 \to (1)$$

Also, $16\sum_{\text{cyc}} a \ge 4\sum_{\text{cyc}} a^2 \to (2)$ $\left(\because 4 \ge a \Rightarrow 4a \ge a^2\ (\because a \ge 0) \text{ and analogs}\right)$ and

moreover, $4\sum_{\text{cyc}} ab \ge a^2b + b^2c + c^2a$ $(\because 4 \ge a, b, c \text{ and } ab, bc, ca \ge 0)$

$$\Rightarrow 8\sum_{\text{cyc}} ab \ge \sum_{\text{cyc}} a^2b + 4\sum_{\text{cyc}} ab \overset{ab,bc,ca \ge 0 \Rightarrow \sum_{\text{cyc}} ab \ge 0}{\ge} \sum_{\text{cyc}} a^2b \to (3)$$

$$\therefore (1) + (2) + (3) \Rightarrow 2\sum_{\text{cyc}} a^3 + 8\sum_{\text{cyc}} ab + 48\sum_{\text{cyc}} a \ge \sum_{\text{cyc}} a^2b + 20\sum_{\text{cyc}} a^2$$

$\Rightarrow (*)$ is true $\therefore 2(x^6 + y^6 + z^6) - x^4y^2 - y^4z^2 - z^4x^2 \le 192\ \forall x, y, z \in [-2, 2]$,

$'' ='' \ (x = 2, y = 2, z = 2)$ or $(x = -2, y = -2, z = -2)$ or $(x = 2, y = 2, z = -2)$

or $(x = 2, y = -2, z = 2)$ or $(x = 2, y = -2, z = -2)$ or $(x = -2, y = 2, z = 2)$

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