ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} \in \mathbb{R}$ and $a^2 + \mathrm{b}^2 + \mathrm{c}^2 \leq 2$, then prove that :

$$2024ca - a\mathrm{b} - \mathrm{bc} \geq -2024$$

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$$2024ca + 2024 - a\mathrm{b} - \mathrm{bc} \overset{2 \geq a^2 + \mathrm{b}^2 + \mathrm{c}^2}{\geq}$$

$$2024ca + 1012(a^2 + \mathrm{b}^2 + \mathrm{c}^2) - \mathrm{b}(\mathrm{c} + a) =$$

$$1012(\mathrm{c}^2 + a^2 + 2ca) - \mathrm{b}(\mathrm{c} + a) + 1012\mathrm{b}^2 = 1012(\mathrm{c} + a)^2 - \mathrm{b}(\mathrm{c} + a) + 1012\mathrm{b}^2$$

$$\geq 0 \left(\begin{matrix} \because 1012(\mathrm{c} + a)^2 - \mathrm{b}(\mathrm{c} + a) + 1012\mathrm{b}^2 \text{ is } a \text{ quadratic polynomial in } ''\mathrm{b}'' \text{ or } ''\mathrm{c} + a'' \\ \text{with discriminant} = (1 - 4.1012^2)(\mathrm{c} + a)^2 \text{ or } (1 - 4.1012^2)\mathrm{b}^2 \leq 0 \end{matrix} \right)$$

$$\text{with equality for } \mathrm{b} = \mathrm{c} + a = 0 \therefore 2024ca - a\mathrm{b} - \mathrm{bc} \geq -2024 \ \forall\, a, \mathrm{b}, \mathrm{c} \in \mathbb{R}$$

$$\Big|\, a^2 + \mathrm{b}^2 + \mathrm{c}^2 \leq 2, ''='' \text{ iff } (a = 1, \mathrm{b} = 0, \mathrm{c} = -1) \text{ or } (a = -1, \mathrm{b} = 0, \mathrm{c} = 1) \text{ (QED)}$$