ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then prove that :

$$27(x+y)^4(y+z)^4(z+x)^4 \ge 4096x^3y^3z^3(x+y+z)^3$$

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$$(x+y)(y+z)(z+x) \overset{?}{\ge} \frac{8}{9}(x+y+z)(xy+yz+zx)$$

$$\Leftrightarrow 9\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - xyz\right) \overset{?}{\ge} 8\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \Leftrightarrow$$

$$\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{?}{\ge} 9xyz \to \text{true} \because \sum_{cyc} x \overset{A-G}{\ge} 3\sqrt[3]{xyz} \text{ and } \sum_{cyc} xy \overset{A-G}{\ge} 3\sqrt[3]{x^2y^2z^2} \Rightarrow$$

$$\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \ge 9xyz \therefore (x+y)(y+z)(z+x) \ge \frac{8}{9}(x+y+z)(xy+yz+zx)$$

$$\Rightarrow 27(x+y)^4(y+z)^4(z+x)^4 \ge \frac{27.4096}{9^4}\left(\sum_{cyc} x\right)^4\left(\sum_{cyc} xy\right)^4$$

$$\ge \frac{27.9xyz.4096}{9^4}\left(\sum_{cyc} x\right)^3\left(\sum_{cyc} xy\right)^3 \overset{A-G}{\ge} \frac{27.9xyz.4096}{9^4}\left(\sum_{cyc} x\right)^3(27x^2y^2z^2)$$

$$\Rightarrow 27(x+y)^4(y+z)^4(z+x)^4 \ge 4096x^3y^3z^3(x+y+z)^3 \ \forall x, y, z > 0,$$

$'' ='' $ iff $x = y = z$ (QED)