ROMANIAN MATHEMATICAL MAGAZINE

If $x, y > 0$ *and* $xy = 1$, then prove that :

$$x^2 + 3x + y^2 + 3y + \frac{9}{x^2 + y^2 + 1} \geq 11$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$x^2 + 3x + y^2 + 3y + \frac{9}{x^2 + y^2 + 1}$$

$$= (x + y)^2 - 2xy + 3(x + y) + \frac{9}{(x + y)^2 - 2xy + 1} \overset{xy=1}{=} t^2 - 2 + 3t + \frac{9}{t^2 - 2 + 1}$$

$$(t = x + y) = \frac{(t^2 - 1)(t^2 - 2 + 3t) + 9}{t^2 - 1} \overset{?}{\geq} 11$$

$$\Leftrightarrow (t^2 - 1)(t^2 - 2 + 3t) + 9 \overset{?}{\geq} 11(t^2 - 1) \Leftrightarrow t^4 + 3t^3 - 14t^2 - 3t + 22 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t - 2)\left((t - 2)(t^2 + 7t + 10) + 9\right) \overset{?}{\geq} 0 \to \text{true} \because t = x + y \overset{A-G}{\geq} 2\sqrt{xy} = 2$$

$$\therefore x^2 + 3x + y^2 + 3y + \frac{9}{x^2 + y^2 + 1} \geq 11$$

$$\forall\, x, y > 0 \,\middle|\, xy = 1, \text{"="} \text{ iff } x = y = 1 \text{ (QED)}$$