ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$, then prove that :

$$(a^{2024}-a^{2022}+3)(b^{2024}-b^{2022}+3)(c^{2024}-c^{2022}+3) \geq 9(ab+bc+ca)$$

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$$a^{2024}-a^{2022}+3 \overset{?}{\geq} a^2+2 \Leftrightarrow a^{2022}(a^2-1) \overset{?}{\geq} a^2-1$$

$$\Leftrightarrow (a^2-1)\left((a^2)^{1011}-1\right) \overset{?}{\geq} 0 \Leftrightarrow (a^2-1)^2\left((a^2)^{1010}+(a^2)^{1009}+\cdots+1\right) \overset{?}{\geq} 0$$

$\rightarrow$ true $\therefore a^{2024}-a^{2022}+3 \geq a^2+2$ and analogs $\Rightarrow$

LHS $\geq (a^2+2)(b^2+2)(c^2+2)$ $\therefore$ it suffices to prove :

$$(a^2+2)(b^2+2)(c^2+2) \boxed{\overset{(\bullet)}{\geq}} 9(ab+bc+ca)$$

$\boxed{\text{Case 1}}$ $a=b=c=0$ and then : LHS of $(\bullet)=8$ and RHS of $(\bullet)=0 \Rightarrow$ LHS $>$ RHS

$\boxed{\text{Case 2}}$ Exactly two among a, b, c equal zero and WLOG we may assume $b=c=0$ and then : LHS of $(\bullet)=4(a^2+2)$ and RHS of $(\bullet)=0 \Rightarrow$ LHS $>$ RHS

$\boxed{\text{Case 3}}$ Exactly one among a, b, c equals zero and WLOG we may assume $a=0$ $(b, c>0)$ and then : LHS of $(\bullet)-$ RHS of $(\bullet)=2(b^2+2)(c^2+2)-9bc$

$$=2b^2c^2+4(b^2+c^2)+8-9bc \overset{A-G}{\geq} 2b^2c^2+8bc+8-9bc$$

$$=(bc-1)^2+b^2c^2+bc+7>0 \Rightarrow \text{LHS} > \text{RHS}$$

$\boxed{\text{Case 4}}$ $a, b, c>0$ and LHS of $(\bullet)=a^2b^2c^2+2+2\sum_{cyc}a^2b^2+4\sum_{cyc}a^2+8$

$$=(a^2b^2c^2+1+1)+2\left((a^2b^2+1)+(b^2c^2+1)+(c^2a^2+1)\right)+4\sum_{cyc}a^2$$

$$\overset{A-G}{\geq} 3\sqrt[3]{a^2b^2c^2}+4\sum_{cyc}ab+4\sum_{cyc}a^2 \overset{?}{\geq} 9(ab+bc+ca)$$

$$\Leftrightarrow 4\sum_{cyc}a^2+3\sqrt[3]{a^2b^2c^2} \boxed{\overset{?}{\underset{(\bullet\bullet)}{\geq}}} 5\sum_{cyc}ab$$

$$\text{Now, } \sum_{cyc}a^2+3\sqrt[3]{a^2b^2c^2} \overset{?}{\geq} 2\sum_{cyc}ab$$

$$\Leftrightarrow \left(\sum_{cyc}a^2\right)^3+27a^2b^2c^2+9\left(\sum_{cyc}a^2\right)^2.\sqrt[3]{a^2b^2c^2}+27\left(\sum_{cyc}a^2\right).\sqrt[3]{a^4b^4c^4}$$

$$\boxed{\overset{?}{\underset{(*)}{\geq}}} 8\left(\sum_{cyc}ab\right)^3$$

Assigning $b+c=x, c+a=y, a+b=z \Rightarrow x+y-z=2c>0, y+z-x=2a>0$ and $z+x-y=2b>0 \Rightarrow x+y>z, y+z>x, z+x>y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2 s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc} (s - x)(s - y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via } (1),(3)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

$$\text{Via } (1), (2) \text{ and } (4), \text{LHS of } (*) \geq (s^2 - 8Rr - 2r^2)^3$$

$$+27r^4s^2 + 9(s^2 - 8Rr - 2r^2)^2 . \sqrt[3]{r^4 s^2} + 27(s^2 - 8Rr - 2r^2). \sqrt[3]{r^8 s^4} \overset{\text{Mitrinovic}}{\geq}$$

$$(s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 9(s^2 - 8Rr - 2r^2)^2 . \sqrt[3]{r^4 . 27r^2}$$

$$+27(s^2 - 8Rr - 2r^2). \sqrt[3]{r^8 . 729r^4}$$

$$= (s^2 - 8Rr - 2r^2)^3 + 27r^4s^2 + 27r^2(s^2 - 8Rr - 2r^2)^2 + 243r^4(s^2 - 8Rr - 2r^2)$$

$$\overset{?}{\geq} 8\left(\sum_{cyc} ab\right)^3 \overset{\text{via } (3)}{=} 8(4Rr + r^2)^3$$

$$\Leftrightarrow s^6 - (24Rr - 21r^2)s^4 + r^2(192R^2 - 336Rr + 174r^2)s^2$$

$$-r^3(1024R^3 - 960R^2r + 1272Rr^2 + 394r^3) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and}$$

$$\because (s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{it suffices to prove :}$$

$$: \text{LHS of } (**) \geq (s^2 - 16Rr + 5r^2)^3 \Leftrightarrow$$

$$(8R + 2r)s^4 - r(192R^2 - 48Rr - 33r^2)s^2$$

$$+r^2(1024R^3 - 960R^2r - 24Rr^2 - 173r^3) \boxed{\overset{(***)}{\geq}} 0$$

$$\text{and} \because (8R + 2r)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (8R + 2r)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (64R^2 + 32Rr + 13r^2)s^2 \boxed{\overset{(****)}{\geq}} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3)$$

$$\text{Finally}, (64R^2 + 32Rr + 13r^2)s^2 \overset{\text{Gerretsen}}{\geq} (64R^2 + 32Rr + 13r^2)(16Rr - 5r^2)$$

$$\overset{?}{\geq} r(1024R^3 + 192R^2r - 96Rr^2 + 223r^3) \Leftrightarrow 144r^3(R - 2r) \overset{?}{\geq} 0 \to \text{true via Euler}$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} a^2 + 3\sqrt[3]{a^2b^2c^2} \geq 2\sum_{cyc} ab$$

$$\Rightarrow 4\sum_{cyc} a^2 + 3\sqrt[3]{a^2b^2c^2} \geq 2\sum_{cyc} ab + 3\sum_{cyc} a^2 \geq 5\sum_{cyc} ab \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \text{combining all cases}, (\bullet) \text{ is true } \forall\, a, b, c \geq 0$$

$$\therefore (a^{2024} - a^{2022} + 3)(b^{2024} - b^{2022} + 3)(c^{2024} - c^{2022} + 3) \geq 9(ab + bc + ca)$$

$$\forall\, a, b, c \geq 0, \text{"="} \text{ iff } a = b = c = 1 \text{ (QED)}$$