ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 = 3$, then prove that :

$$\left(5a + \frac{2}{b+c}\right)^3 + \left(5b + \frac{2}{c+a}\right)^3 + \left(5c + \frac{2}{a+b}\right)^3 \geq 648$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(5a + \frac{2}{b+c}\right)^3 = \sum_{cyc}\left(a+a+a+a+a+\frac{2}{b+c}\right)^3 \overset{A-G}{\geq}$$

$$\sum_{cyc}\left(6.\sqrt[6]{\frac{2a^5}{b+c}}\right)^3 = 216.\sqrt{2}.\sum_{cyc}\sqrt{\frac{a^8}{a^3b + a^3c}} = 216.\sqrt{2}.\sum_{cyc}\frac{a^4}{\sqrt{a^3b + a^3c}} \overset{Bergstrom}{\geq}$$

$$216.\sqrt{2}.\frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc}\sqrt{a^3b + a^3c}} \overset{CBS}{\geq} 216.\sqrt{2}.\frac{\left(\sum_{cyc} a^2\right)^2}{\sqrt{3}.\sqrt{\sum_{cyc}(a^3b + ab^3)}} \overset{a^2+b^2+c^2=3}{=}$$

$$648.\sqrt{2}.\frac{\left(\sum_{cyc} a^2\right)^2}{\sqrt{3}.\left(\sum_{cyc} a^2\right).\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) - abc\sum_{cyc} a}} \overset{?}{\geq} 648$$

$$\Leftrightarrow 2\left(\sum_{cyc} a^2\right)^2 \boxed{\overset{?}{\underset{(*)}{\geq}}} 3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right) - 3abc\sum_{cyc} a$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

$$\text{so } 2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$$

$$\therefore abc = r^2s \to (2) \text{ and such substitutions} \Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{via\ (1),(3)}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4)$$

Via $(1), (2), (3)$ and $(4), (*) \Leftrightarrow$

$$2(s^2 - 8Rr - 2r^2)^2 - 3(s^2 - 8Rr - 2r^2)(4Rr + r^2) + 3r^2s^2 \boxed{\overset{(**)}{\geq}} 0 \text{ and}$$

$\because (s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\geq} 0 \therefore$ in order to prove $(**)$, it suffices to prove :

$$\text{LHS of } (**) \geq (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (5R - 7r)s^2 \boxed{\overset{(***)}{\geq}} r(72R^2 - 108Rr + 9r^2)$$

$$\text{Now, } (5R - 7r)s^2 \overset{Gerretsen}{\geq} (5R - 7r)(16Rr - 5r^2) \overset{?}{\geq} r(72R^2 - 108Rr + 9r^2)$$

$$\Leftrightarrow 8R^2-29Rr+26r^2 \overset{?}{\geq} 0 \Leftrightarrow (R-2r)(8R-13r) \overset{?}{\geq} 0 \rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \left(5a+\frac{2}{b+c}\right)^3+\left(5b+\frac{2}{c+a}\right)^3+\left(5c+\frac{2}{a+b}\right)^3$$

$$\geq 648\ \forall\ a,b,c>0 \,\middle|\, a^2+b^2+c^2=3, ''='' \text{ iff } a=b=c=1 \text{ (QED)}$$

Solution 2 by Tapas Das-India

$$Note: Vasc's\ inequality\ (a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a)$$

$$\left(5a+\frac{2}{b+c}\right)^3+\left(5b+\frac{2}{c+a}\right)^3+\left(5c+\frac{2}{a+b}\right)^3 \overset{AM-GM}{\geq}$$

$$\geq \left(\sum (6\sqrt[6]{\frac{2a^5}{b+c}}\right)^3 = 216.\sqrt{2}\left(\sum \frac{a^{\frac{5}{2}}}{(b+c)^{\frac{1}{2}}}\right) =$$

$$= 216.\sqrt{2}\left(\sum \frac{a^{\frac{5}{2}}a^{\frac{3}{2}}}{a^{\frac{3}{2}}(b+c)^{\frac{1}{2}}}\right) = 216.\sqrt{2}\left(\sum \frac{a^4}{\sqrt{a^3(b+c)}}\right) \overset{Bergstrom\,\&\,CBS}{\geq}$$

$$\geq 216.\sqrt{2}\frac{(\sum a^2)^2}{\left(3\left((a^3b+b^3c+c^3a)+(a^3c+c^3b+b^3a)\right)\right)^{\frac{1}{2}}} \overset{Vasc}{\geq}$$

$$216.\sqrt{2}\frac{(\sum a^2)^2}{\sqrt{2}\,(\sum a^2)} = 216.\left(\sum a^2\right) = 216\,.\,3 = 648$$