ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, then prove that :

$$\frac{x^2 - z^2}{y + z} + \frac{z^2 - y^2}{x + y} + \frac{y^2 - x^2}{z + x} \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x^2 - z^2}{y + z} + \frac{z^2 - y^2}{x + y} + \frac{y^2 - x^2}{z + x}$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \sum_{cyc} \left((x^2 - z^2)(x + y)(z + x)\right)$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \sum_{cyc} \left((x^2 - z^2)\left(\sum_{cyc} xy + x^2\right)\right)$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \left(\left(\sum_{cyc} xy\right)(x^2 - z^2 + z^2 - y^2 + y^2 - x^2) + \sum_{cyc} \left(x^2(x^2 - z^2)\right)\right)$$

$$= \frac{1}{(x + y)(y + z)(z + x)} \cdot \left(\sum_{cyc} x^4 - \sum_{cyc} x^2y^2\right) \geq 0$$

$$\therefore \frac{x^2 - z^2}{y + z} + \frac{z^2 - y^2}{x + y} + \frac{y^2 - x^2}{z + x} \geq 0 \ \forall\, x, y, z > 0, '' ='' \text{ iff } x = y = z \text{ (QED)}$$

Solution 2 by Mirsadix Muzefferov-Azerbaijan

Let $x + y = m \ ; \ y + z = n \ ; \ x + z = p$

Then

$$\begin{cases} x + y = m \\ y + z = n \\ z + x = p \end{cases} \Rightarrow \begin{cases} x = \dfrac{m + p - n}{2} \\ y = \dfrac{m + n - p}{2} \\ z = \dfrac{n + p - m}{2} \end{cases} \Rightarrow \begin{cases} x - z = m - n \\ y - z = n - p \\ z - y = p - m \end{cases}$$

The given inequality will be as follows:

$$\frac{p(m - n)}{n} + \frac{n(p - m)}{m} + \frac{m(n - p)}{p} \geq 0 \Rightarrow \frac{pm}{n} + \frac{np}{m} + \frac{mn}{p} \geq m + n + p \ \ (*)$$

Prove that $(*)$**:**

$$\begin{cases} m\left(\frac{p}{n}+\frac{n}{p}\right) \geq 2m \\ n\left(\frac{m}{p}+\frac{p}{m}\right) \geq 2n \quad (**) \\ p\left(\frac{n}{m}+\frac{m}{n}\right) \geq 2p \end{cases}$$

$(**)$ $\textit{let's summarize side by side}$

$$\frac{pm}{n}+\frac{np}{m}+\frac{mn}{p} \geq m+n+p$$

Equality holds iff x=y=z .