ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geq 1$, then:

$$a+b+c \geq ab+bc+ca$$

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$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geq 1 \Rightarrow \sum_{cyc}\frac{1+a+b-(a+b)}{a+b+1} \geq 1$$

$$\Rightarrow 3 \geq 1+\sum_{cyc}\frac{a+b}{a+b+1} \Rightarrow 2 \geq \sum_{cyc}\frac{(a+b)^2}{(a+b)^2+(a+b)}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\left(2\sum_{cyc}a\right)^2}{2\sum_{cyc}a^2+2\sum_{cyc}ab+2\sum_{cyc}a}$$

$$\Rightarrow \sum_{cyc}a^2+\sum_{cyc}ab+\sum_{cyc}a \geq \left(\sum_{cyc}a\right)^2 = \sum_{cyc}a^2+2\sum_{cyc}ab \Rightarrow \sum_{cyc}a \geq \sum_{cyc}ab$$

$$\therefore a+b+c \geq ab+bc+ca$$

$$\forall\, a, b, c > 0 \,\Big|\, \frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1} \geq 1 \text{ (QED)}$$