ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$, then prove that :

$$32a^2b^2(a^3+b^3) \le (a^2+b^2)(a+b)^5$$

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$$(a^2+b^2+2ab)^2 \overset{A-G}{\ge} 4(a^2+b^2)(2ab) \Rightarrow (a+b)^4 \ge 8ab(a^2+b^2)$$

$$\Rightarrow (a^2+b^2)(a+b)^5 \ge 8ab(a^2+b^2)(a+b)(a^2+b^2) \overset{?}{\ge} 32a^2b^2(a^3+b^3)$$

$$= 32a^2b^2(a+b)(a^2+b^2-ab) \Leftrightarrow (a^2+b^2)^2 \overset{?}{\ge} 4ab(a^2+b^2-ab)$$

$$\Leftrightarrow (a^2+b^2)^2 - 4ab(a^2+b^2) + 4a^2b^2 \overset{?}{\ge} 0 \Leftrightarrow (a^2+b^2-2ab)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a-b)^4 \overset{?}{\ge} 0 \to \text{true} \therefore 32a^2b^2(a^3+b^3) \le (a^2+b^2)(a+b)^5$$

$$\forall\, a, b > 0, '' ='' \text{ iff } a = b \text{ (QED)}$$