ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$, then prove that :

$$(a^2+2)(\mathrm{b}^2+2)(\mathrm{c}^2+2) \geq 3(a+\mathrm{b}+\mathrm{c})^2+(a\mathrm{bc}-1)^2$$

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$$(a^2+2)(\mathrm{b}^2+2)(\mathrm{c}^2+2)-3(a+\mathrm{b}+\mathrm{c})^2-(a\mathrm{bc}-1)^2$$
$$= a^2\mathrm{b}^2\mathrm{c}^2+8+4\sum_{\mathrm{cyc}} a^2+2\sum_{\mathrm{cyc}} a^2\mathrm{b}^2-3\sum_{\mathrm{cyc}} a^2-6\sum_{\mathrm{cyc}} a\mathrm{b}-a^2\mathrm{b}^2\mathrm{c}^2-1+2a\mathrm{bc}$$
$$= (a\mathrm{bc}+a\mathrm{bc}+1)+2\left((a^2\mathrm{b}^2+1)+(\mathrm{b}^2\mathrm{c}^2+1)+(\mathrm{c}^2a^2+1)\right)+\sum_{\mathrm{cyc}} a^2-6\sum_{\mathrm{cyc}} a\mathrm{b}$$
$$\overset{\mathrm{A-G}}{\geq} 3\sqrt[3]{a^2\mathrm{b}^2\mathrm{c}^2}+2(2a\mathrm{b}+2\mathrm{bc}+2\mathrm{c}a)+\sum_{\mathrm{cyc}} a^2-6\sum_{\mathrm{cyc}} a\mathrm{b} \overset{?}{\geq} 0$$
$$\Leftrightarrow \sum_{\mathrm{cyc}} a^2+3\sqrt[3]{a^2\mathrm{b}^2\mathrm{c}^2} \overset{?}{\geq} 2\sum_{\mathrm{cyc}} a\mathrm{b}$$
$$\Leftrightarrow \left(\sum_{\mathrm{cyc}} a^2\right)^3+27a^2\mathrm{b}^2\mathrm{c}^2+9\left(\sum_{\mathrm{cyc}} a^2\right)^2.\sqrt[3]{a^2\mathrm{b}^2\mathrm{c}^2}+27\left(\sum_{\mathrm{cyc}} a^2\right).\sqrt[3]{a^4\mathrm{b}^4\mathrm{c}^4}$$
$$\boxed{\underset{(*)}{\overset{?}{\geq}}}\, 8\left(\sum_{\mathrm{cyc}} a\mathrm{b}\right)^3$$

Assigning $\mathrm{b}+\mathrm{c}=x, \mathrm{c}+a=\mathrm{y}, a+\mathrm{b}=\mathrm{z} \Rightarrow x+\mathrm{y}-\mathrm{z}=2\mathrm{c}>0, y+z-x=2a > 0$ and $\mathrm{z}+x-\mathrm{y}=2\mathrm{b}>0 \Rightarrow x+\mathrm{y}>z, \mathrm{y}+\mathrm{z}>x, \mathrm{z}+x>y \Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (say);

$$\text{so } 2\sum_{\mathrm{cyc}} a=\sum_{\mathrm{cyc}} x=2\mathrm{s} \Rightarrow \sum_{\mathrm{cyc}} a=\mathrm{s} \rightarrow (1) \Rightarrow a=\mathrm{s}-x, \mathrm{b}=\mathrm{s}-\mathrm{y}, \mathrm{c}=\mathrm{s}-\mathrm{z}$$
$$\therefore a\mathrm{bc}=\mathrm{r}^2\mathrm{s} \rightarrow (2) \text{ and such substitutions} \Rightarrow \sum_{\mathrm{cyc}} a\mathrm{b}=\sum_{\mathrm{cyc}}(\mathrm{s}-x)(\mathrm{s}-\mathrm{y}) \Rightarrow$$
$$\sum_{\mathrm{cyc}} a\mathrm{b}=4\mathrm{Rr}+\mathrm{r}^2 \rightarrow (3), \sum_{\mathrm{cyc}} a^2=\left(\sum_{\mathrm{cyc}} a\right)^2-2\sum_{\mathrm{cyc}} a\mathrm{b} \overset{\text{via } (1),(3)}{=} \mathrm{s}^2-2(4\mathrm{Rr}+\mathrm{r}^2)$$
$$\Rightarrow \sum_{\mathrm{cyc}} a^2=\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2 \rightarrow (4)$$

Via (1), (2) and (4), LHS of $(*) \geq$

$$(\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2)^3+27\mathrm{r}^4\mathrm{s}^2+9(\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2)^2.\sqrt[3]{\mathrm{r}^4\mathrm{s}^2}$$
$$+27(\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2).\sqrt[3]{\mathrm{r}^8\mathrm{s}^4} \overset{\text{Mitrinovic}}{\geq}$$
$$(\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2)^3+27\mathrm{r}^4\mathrm{s}^2+9(\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2)^2.\sqrt[3]{\mathrm{r}^4.27\mathrm{r}^2}$$
$$+27(\mathrm{s}^2-8\mathrm{Rr}-2\mathrm{r}^2).\sqrt[3]{\mathrm{r}^8.729\mathrm{r}^4}$$

$$= \left(s^2-8Rr-2r^2\right)^3+27r^4s^2+27r^2\left(s^2-8Rr-2r^2\right)^2+243r^4\left(s^2-8Rr-2r^2\right)$$

$$\overset{?}{\geq} 8\left(\sum_{cyc} ab\right)^3 \overset{\text{via } (3)}{=} 8\left(4Rr+r^2\right)^3$$

$$\Leftrightarrow s^6-\left(24Rr-21r^2\right)s^4+r^2\left(192R^2-336Rr+174r^2\right)s^2$$

$$-r^3\left(1024R^3-960R^2r+1272Rr^2+394r^3\right) \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and}$$

$$\because \left(s^2-16Rr+5r^2\right)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**),$$

$$\text{it suffices to prove : LHS of } (**) \geq \left(s^2-16Rr+5r^2\right)^3$$

$$\Leftrightarrow (8R+2r)s^4-r\left(192R^2-48Rr-33r^2\right)s^2$$

$$+r^2\left(1024R^3-960R^2r-24Rr^2-173r^3\right) \boxed{\overset{(***)}{\geq}} 0 \text{ and } \because$$

$$(8R+2r)\left(s^2-16Rr+5r^2\right)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (***),$$

$$\text{it suffices to prove : LHS of } (***) \geq (8R+2r)\left(s^2-16Rr+5r^2\right)^2$$

$$\Leftrightarrow \left(64R^2+32Rr+13r^2\right)s^2 \boxed{\overset{(****)}{\geq}} r\left(1024R^3+192R^2r-96Rr^2+223r^3\right)$$

$$\text{Finally, } \left(64R^2+32Rr+13r^2\right)s^2 \overset{\text{Gerretsen}}{\geq} \left(64R^2+32Rr+13r^2\right)\left(16Rr-5r^2\right)$$

$$\overset{?}{\geq} r\left(1024R^3+192R^2r-96Rr^2+223r^3\right) \Leftrightarrow 144r^3(R-2r) \overset{?}{\geq} 0 \rightarrow \text{true via Euler}$$

$$\Rightarrow (****) \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \left(a^2+2\right)\left(b^2+2\right)\left(c^2+2\right)$$

$$\geq 3(a+b+c)^2+(abc-1)^2 \ \forall\, a,b,c>0, \text{"="} \text{ iff } a=b=c=1 \text{ (QED)}$$