ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $ab^2c^3>\frac{1}{7}, bc^2a^3>\frac{1}{7}. ca^2b^3>\frac{1}{7}$, then prove that :

$$\frac{a^7}{7ab^2c^3-1}+\frac{b^7}{7bc^2a^3-1}+\frac{c^7}{7ca^2b^3-1}\ge\frac{1}{2}$$

Proposed by Fazil Maharramov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } ab^2c^3>\frac{1}{7}, bc^2a^3>\frac{1}{7}. ca^2b^3>\frac{1}{7}$$

$$\therefore \frac{a^7}{7ab^2c^3-1}+\frac{b^7}{7bc^2a^3-1}+\frac{c^7}{7ca^2b^3-1}$$

$$=\frac{a^9}{7a^3b^2c^3-a^2}+\frac{b^9}{7b^3c^2a^3-b^2}+\frac{c^9}{7c^3a^2b^3-c^2}$$

$$\overset{\text{Holder}}{\ge}\frac{\left(\sum_{cyc}a^3\right)^3}{21a^2b^2c^2\sum_{cyc}ab-3\sum_{cyc}a^2}\overset{?}{\ge}\frac{1}{2}$$

$$\Leftrightarrow 2\left(\sum_{cyc}a^3\right)^3+3\sum_{cyc}a^2\underset{(*)}{\overset{?}{\ge}}21a^2b^2c^2\sum_{cyc}ab$$

$$\text{Now, Power}-\text{Mean inequality}\Rightarrow\left(\frac{\sum_{cyc}a^3}{3}\right)^{\frac{1}{3}}\ge\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{1}{2}}$$

$$\Rightarrow\sum_{cyc}a^3\ge3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}\Rightarrow2\left(\sum_{cyc}a^3\right)^3\ge18\left(\frac{\sum_{cyc}a^2}{3}\right)^3.3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}$$

$$=2\left(\sum_{cyc}a^2\right)^3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}\to(1)$$

$$\text{Again, }21a^2b^2c^2\sum_{cyc}ab\overset{A-G}{\le}\frac{7}{9}\left(\sum_{cyc}a^2\right)^4\to(2)\ \therefore(1),(2)\Rightarrow\text{in order}$$

$$\text{to prove }(*),\text{it suffices to prove : }2\left(\sum_{cyc}a^2\right)^3\left(\frac{\sum_{cyc}a^2}{3}\right)^{\frac{3}{2}}+3\sum_{cyc}a^2\ge\frac{7}{9}\left(\sum_{cyc}a^2\right)^4$$

$$\Leftrightarrow 2.27t^6.t^3+9t^2\ge\frac{7}{9}.81t^8\left(t=\sqrt{\frac{\sum_{cyc}a^2}{3}}\right)\Leftrightarrow t^2(6t^7-7t^6+1)\ge0$$

$$\Leftrightarrow(t-1)^2(6t^5+5t^4+4t^3+3t^2+2t+1)\ge0\to\text{true}\Rightarrow(*)\text{ is true}$$

$\therefore \frac{a^7}{7ab^2c^3-1}+\frac{b^7}{7bc^2a^3-1}+\frac{c^7}{7ca^2b^3-1}\geq\frac{1}{2}\ \forall\ a,b,c>0 \mid ab^2c^3>\frac{1}{7},$
$bc^2a^3>\frac{1}{7}.ca^2b^3>\frac{1}{7},''=''$ iff $a=b=c$ and $\sqrt{\frac{\sum_{cyc}a^2}{3}}=1\Rightarrow\sum_{cyc}a^2=3$
$\therefore ''=''$ iff $a=b=c=1$ (QED)