ROMANIAN MATHEMATICAL MAGAZINE

If $a > 2, b > \frac{2}{3}, c > 4$, *then prove that*:

$$\frac{a^3}{(9b-6)^2} + \frac{3b^3}{(c-4)^2} + \frac{c^3}{(3a-6)^2} \geq 6$$

Proposed by Fazil Maharramov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } a > 2, b > \frac{2}{3}, c > 4 \therefore \frac{a^3}{(9b-6)^2} + \frac{3b^3}{(c-4)^2} + \frac{c^3}{(3a-6)^2}$$

$$= \frac{a^3}{(9b-6)^2} + \frac{27b^3}{(3c-12)^2} + \frac{c^3}{(3a-6)^2} \overset{\text{Radon}}{\geq} \frac{(a+3b+c)^3}{9(a+3b+c-8)^2} = \frac{x^3}{9(x-8)^2} \overset{?}{\geq} 6$$

$$(x = a+3b+c) \Leftrightarrow x^3 - 54x^2 + 864x - 3456 \overset{?}{\geq} 0 \Leftrightarrow (x-6)(x-24)^2 \overset{?}{\geq} 0$$

$$\to \text{true} \because a+3b+c > 2+3\cdot\frac{2}{3}+4 = 8 \Rightarrow x-6 > 2 > 0, \text{equality iff}$$

$$\frac{a}{3b-2} = \frac{3b}{c-4} = \frac{c}{a-2} = \frac{a+3b+c}{a+3b+c-8} = \frac{24}{24-8} = \frac{3}{2} \Rightarrow \text{iff } 2a = 9b-6 \to (\text{i}),$$

$$6b = 3c-12 \to (\text{ii}), 2c = 3a-6 \to (\text{iii}) \therefore (\text{i}) \text{ and } (\text{iii}) \Rightarrow 6a = 27b-18 \text{ and}$$

$$6a = 4c+12 \Rightarrow 27b-18 = 4c+12 \Rightarrow 54b = 8c+60 \to (1) \text{ and also},$$

$$\text{via } (\text{ii}), 54b = 27c-108 \to (2) \therefore (1),(2) \Rightarrow 8c+60 = 27c-108 \Rightarrow c \overset{(*)}{=} \frac{168}{19}$$

$$\text{Putting } c = \frac{168}{19} \text{ in (ii) and (iii) respectively, we get: } b = \frac{46}{19} \text{ and } a = \frac{150}{19}$$

$$\therefore \frac{a^3}{(9b-6)^2} + \frac{3b^3}{(c-4)^2} + \frac{c^3}{(3a-6)^2} \geq 6 \ \forall\, a > 2, b > \frac{2}{3}, c > 4,$$

$$'' ='' \text{ iff } a = \frac{150}{19}, b = \frac{46}{19}, c = \frac{168}{19} \text{ (QED)}$$