ROMANIAN MATHEMATICAL MAGAZINE

If $a^2b > 2, b^2c > 2, c^2a > 2$, then prove that :

$$\frac{a^4}{a^2b-2}+\frac{b^4}{b^2c-2}+\frac{c^4}{c^2a-2}\geq 8$$

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$$\text{Since } a^2b > 2, b^2c > 2, c^2a > 2$$

$$\therefore \frac{a^4}{a^2b-2}+\frac{b^4}{b^2c-2}+\frac{c^4}{c^2a-2}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} a^2b-6} \overset{?}{\geq} 8 \Leftrightarrow \left(\sum_{cyc} a^2\right)^2 + 48 \underset{(*)}{\overset{?}{\geq}} 8\sum_{cyc} a^2b$$

$$\text{Now, } 8\sum_{cyc} a^2b \overset{\text{CBS}}{\leq} 8.\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} \leq 8.\sqrt{\left(\sum_{cyc} a^2\right).\frac{\left(\sum_{cyc} a^2\right)^2}{3}}$$

$$\overset{?}{\leq} \left(\sum_{cyc} a^2\right)^2 + 48 \Leftrightarrow 8.\left(\sum_{cyc} a^2\right).\sqrt{\frac{\sum_{cyc} a^2}{3}} \overset{?}{\leq} \left(\sum_{cyc} a^2\right)^2 + 48$$

$$\Leftrightarrow 8t.3t^2 \overset{?}{\leq} 9t^4 + 48 \left(t = \sqrt{\frac{\sum_{cyc} a^2}{3}}\right) \Leftrightarrow 3t^4 - 8t^3 + 16 \overset{?}{\geq} 0$$

$$\Leftrightarrow (t-2)^2(3t^2+4t+4) \overset{?}{\geq} 0 \to \text{true} \because t = \sqrt{\frac{\sum_{cyc} a^2}{3}} > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{a^4}{a^2b-2}+\frac{b^4}{b^2c-2}+\frac{c^4}{c^2a-2} \geq 8 \ \forall\, a^2b > 2, b^2c > 2, c^2a > 2, '' ='' \text{ iff } a = b = c$$

$$\text{and } \sqrt{\frac{\sum_{cyc} a^2}{3}} = 2 \Rightarrow \sum_{cyc} a^2 = 12 \therefore '' ='' \text{ iff } a = b = c = 2 \ (\text{QED})$$