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If $a,b,c>0$ then:

$$ab\cdot 2^{\frac{c}{b}}+bc\cdot 2^{\frac{a}{c}}+ca\cdot 2^{\frac{b}{a}}\geq 2(ab+bc+ca)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tapas Das-India

$$\frac{ab\left(2^{\frac{c}{b}}\right)+bc\left(2^{\frac{a}{c}}\right)+ca\left(2^{\frac{b}{a}}\right)}{ab+bc+ca}\geq\left[\left(2^{\frac{c}{b}}\right)^{ab}\cdot\left(2^{\frac{a}{c}}\right)^{bc}\cdot\left(2^{\frac{b}{a}}\right)^{ca}\right]^{\frac{1}{ab+bc+ca}}=$$

$$=\left(2^{ac}\cdot 2^{ab}\cdot 2^{bc}\right)^{\frac{1}{ab+bc+ca}}=2^{(ab+bc+ca)\cdot\frac{1}{(ab+bc+ca)}}=2$$

$$\therefore ab\,2^{\frac{c}{b}}+bc\cdot 2^{\frac{a}{c}}+ca\cdot 2^{\frac{b}{a}}\geq 2(ab+bc+ca)$$

Equality holds for $a=b=c$.

Solution 2 by Khaled Abd Imouti-Syria

$$ab\cdot 2^{\frac{c}{b}}+bc\cdot 2^{\frac{a}{c}}+ca\cdot 2^{\frac{b}{a}}\overset{?}{\geq} 2(ab+bc+ca)$$

$$e_1=\left(\frac{ab}{ab+bc+ca}\right)\left(2^{\frac{c}{b}}\right)+\left(\frac{bc}{ab+bc+ca}\right)\left(2^{\frac{a}{c}}\right)+\left(\frac{ca}{ab+bc+ca}\right)\left(2^{\frac{b}{a}}\right)\overset{?}{\geq} 2$$

By using weighted AM-GM inequality

$$e_1=\left(\left(2^{\frac{c}{b}}\right)^{ab}\cdot\left(2^{\frac{a}{c}}\right)^{bc}\cdot\left(2^{\frac{b}{a}}\right)^{ca}\right)^{\frac{1}{ab+bc+ca}}$$

$$e_1\geq 2^{(ca+bc+ab)\cdot\frac{1}{(ab+bc+ca)}}$$

$$e_1\geq 2$$

Equality holds for $a=b=c$.