ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ then:

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}\ge\sqrt{2}(ab+bc+ca)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}$$
$$=a|c+ib|+b|a+ic|+c|b+ia|=|ac+iab|+|ba+ibc|+|bc+iac|$$
$$\ge|ac+ba+bc+i(ab+bc+ac)|=(ab+bc+ca)|1+i|$$
$$=\sqrt{2}(ab+bc+ca)$$

Solution 2 by Tapas Das-India

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}$$
$$=\sqrt{a^2c^2+a^2b^2}+\sqrt{b^2a^2+b^2c^2}+\sqrt{c^2b^2+c^2a^2}$$
$$\overset{Minkowski}{\ge}\sqrt{(ac+ba+cb)^2+(ab+bc+ca)^2}$$
$$=\sqrt{2(ab+bc+ca)^2}=\sqrt{2}(ab+bc+ca)$$

Solution 3 by Khaled Abd Imouti-Syria

$$a\sqrt{c^2+b^2}+b\sqrt{a^2+c^2}+c\sqrt{b^2+a^2}\ge\sqrt{2}(ab+bc+ca)$$
$$\sqrt{a^2c^2+a^2b^2}+\sqrt{b^2a^2+c^2b^2}+\sqrt{c^2b^2+c^2a^2}\overset{?}{\ge}\sqrt{2}(ab+bc+ca)$$
$$\underbrace{\sqrt{\frac{a^2c^2+a^2b^2}{2}}+\sqrt{\frac{b^2a^2+c^2b^2}{2}}+\sqrt{\frac{c^2b^2+c^2a^2}{2}}}_{e_1}\ge ab+bc+ca$$

By using AM-GM

$$e_1\ge\frac{ac+ab+bc+cb+cb+ca}{2}=\frac{2(ac+ab+bc)}{2}$$
$$e_1\ge ac+ab+bc$$